Introduction

This is the appendix to a paper describing an application of the Analytic Hierarchy Process (AHP) to a streamlined, environmental life-cycle matrix assessment of two aluminum anodizing processes. Specifically this appendix provides a detailed look at the analyses of the two anodizing processes. There are three parts to the appendix: a list of heuristic rules, the AHP matrix computations and a description of the necessary matrix manipulations using matrix notation. Please consult the paper for specifics about AHP and its value.

Part 1 - Heuristic Rules

It is important to note that ranking is more of an art than a science. This section describes the heuristics used to evaluate a manufacturing process matrix. When doing a ranking the user needs to bear in mind that the objective is to design or select a process with the least adverse environmental impact. Two factors are embedded in the objective: the ability to design or influence the design and environmental impact. As in any ranking scheme there is always a danger that the user will have decided in advance the preferred process and will shape the rankings to achieve this result. This may occur at a subconscious level. Consequently, the user should be careful not to weigh the different life-cycle stages to prejudice the outcome in a pre-selected manner. The user should take into account information based on existing processes and the relative importance of various life-cycle steps. At the same time, the user should be careful not to ignore other factors, because alternative processes may have environmental effects elsewhere.

The following heuristic rules may be of use. It is important to keep in mind that these rules are meant to be guidelines, not hard and fast, no-exception rules.

1. Generally, the infrastructure life-cycle stage (lc1 in the notation of Figure 1 of this appendix) will be of lesser concern for processes that are chemical in nature and that operate at or near ambient temperatures and pressures. For mechanical processes or processes where the equipment is massive, this stage will have increased importance. Factors that would also increase relative importance would be the need to construct or significantly modify facilities, especially if substantial land-use is involved, or to use substantial quantities (mass) of equipment. Ranking would increase if the equipment would be designed, as opposed to off-the-shelf. Importance can be expected to be lower if existing equipment can be utilized. If equipment must be replaced frequently, importance of this life-cycle stage would also increase.

2. The manufacturing steps (lc2, lc3 and lc4), including the process under analysis (PUA) (lc3), prior-PUA (lc2) and post-PUA (lc4), will be of relatively high importance because of the ability to influence design, especially the PUA step, which is the focus of the LCA. Ranking would increase for steps that require maintenance of temperatures that differ significantly from ambient air temperatures, with greater temperature difference having the greater influence on ranking. Steps that use substantial quantities of chemicals or resources such as water or fuel would have increased importance. In order to focus attention on the PUA, the user would generally rank the PUA higher than the other manufacturing steps unless other factors suggest otherwise.

3. The disposal (or process termination) step (lc5) will also have a lower ranking in general for processes, facilities and equipment that have a long useful life. If equipment or facilities will accumulate residues from the process, then a higher ranking may be appropriate. If the facility or equipment is large and has no likely use after termination, the ranking would increase.

4. The product aspect is potentially quite significant, although it is frequently overlooked. To the extent that products (or parts or components) that pass through the PUA are long-lived, relative importance of this life-cycle stage (lc6) will increase. If the products are massive, this too will increase importance. If the product consumes energy or is part of a mobile piece of equipment, increased importance will be associated with this life-cycle step. If the product requires maintenance, then importance will increase, with more maintenance resulting in greater importance. If the process (PUA) results in material being added to the product, then this life-cycle step will receive added ranking. If the product can be expected to operate in sensitive environments (such as near domestic water supplies or in smoggy locales), then importance will increase.

Part 2 - AHP Matrix Computations

The streamlined LCA matrix for a process is shown in Figure 1.

	Non-hazardous Materials Choice (NH)	Hazardous Materials Choice (H)	Energy (E)	Solid Residues (S)	Liquid Residues (L)	Gaseous Residues (G)
Process Infrastructure (lc1)						
Pre-process Manufacturing						

(lc2)						
Process Under Analysis (lc3)						
Post-process manufacturing (lc4)						
Process Termination (lc5)						
Products That Pass Through the Process (lc6)						

Figure 1. Streamlined LCA Matrix

From the matrix structure, we create a hierarchy for purposes of AHP. This hierarchical structure is shown in Figure 2.

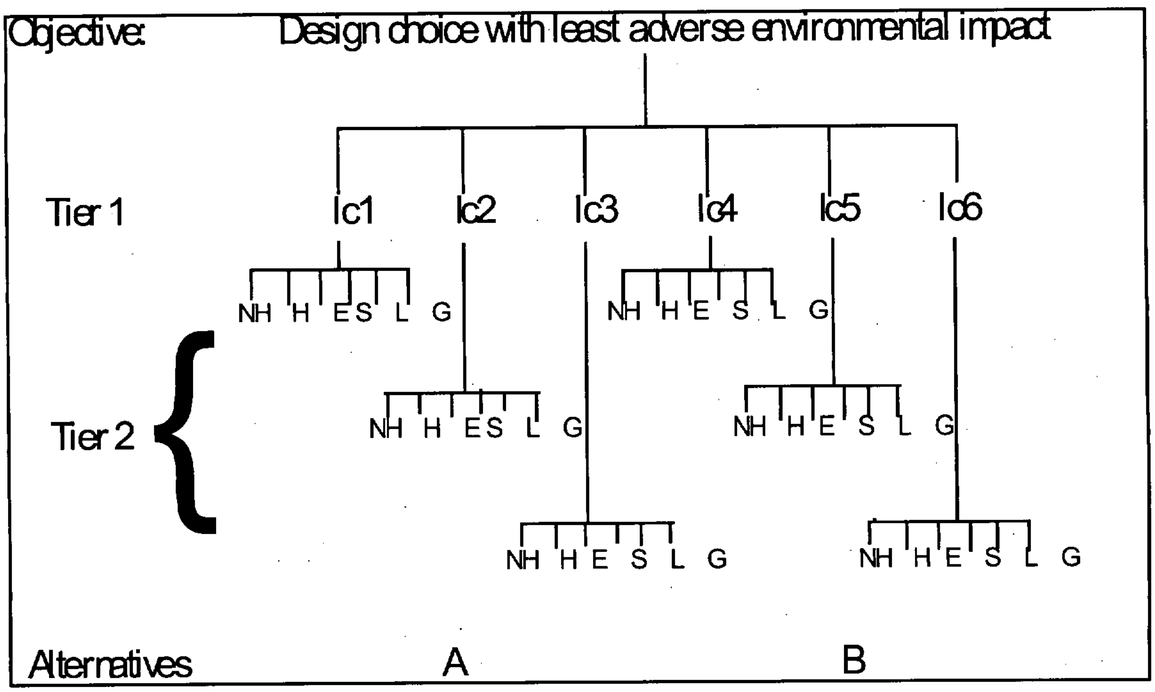

Figure 2

The first two tiers of this hierarchy result in a series of comparison matrices consisting of one 6x6 matrix for the first tier factors and six 6x6 matrices for the second tier factors, as shown in Figures 3 through 9. The matrix entries represent the assessment by the user of the relative importance of the various factors with respect to one another, using the heuristic guidelines from Part 1 of this appendix and the following AHP scoring rules. The entry in any ij cell of an AHP comparison matrix is given by:

a_{ij} = 1 if the row i factor is of equal importance to the column j factor

= 3 if the row i factor is of weak importance compared to the column j factor

= 5 if the row i factor is of strong importance compared to the column j factor

= 7 if the row i factor is of very strong importance compared to the column j factor

= 9 if the row i factor is of absolute importance compared to the column j factor

with:

$a_{ii} = 1$, $a_{ij} = 1/a_{ji}$ and 2, 4, 6, and 8 used as intermediate values to resolve uncertainties or effect compromise.

During the comparison of the life-cycle stages, two factors were kept in mind while doing the pair-wise comparison. These were the designer's ability to influence the life-cycle step and the relative extent of environmental impact of the life-cycle step.

	lc1	lc2	lc3	lc4	lc5	lc6
lc1	1	1/4	1/5	1/4	1/2	1/7
lc2	4	1	1/3	1	3	1/5
lc3	5	3	1	3	4	1/4
lc4	4	1	1/3	1	3	1/5
lc5	2	1/3	1/4	1/3	1	1/7
lc6	7	5	4	5	7	1

Figure 3. Tier 1 Comparison AHP Matrix

Tier 2 AHP Comparison Matrix with Respect to lc1 (PUA Infrastructure)

	NH	H	E	S	L	G
NH	1	2	1/4	1/2	2	2
H	1/2	1	1/3	1/3	1	1
E	4	3	1	1	2	2
S	2	3	1	1	2	1
L	1/2	1	1/2	1/2	1	1
G	1/2	1	1/2	1/2	1	1

Figure 4

Tier 2 AHP Comparison Matrix with Respect to lc2 (pre-PUA Manufacturing)

	NH	H	E	S	L	G
NH	1	1/4	1/4	1/3	1/2	1/4
H	4	1	1	1/2	1/2	1/3
E	4	1	1	2	1	1
S	3	2	1/2	1	1	1/2
L	2	2	1	1	1	1/2
G	4	3	1	2	2	1

Figure 5

Tier 2 AHP Comparison Matrix with Respect to lc3 (PUA)

	NH	H	E	S	L	G
NH	1	1/4	1/4	1/3	1/2	1/4
H	4	1	1	1/2	1/2	1/3
E	4	1	1	2	1	1
S	3	2	1/2	1	1	1/2
L	2	2	1	1	1	1/2
G	4	3	1	2	2	1

Figure 6

Tier 2 AHP Comparison Matrix with Respect to lc4 (post-PUA Manufacturing)

	NH	H	E	S	L	G
NH	1	1/4	1/4	1/3	1/2	1/4
H	4	1	1	1/2	1/2	1/3
E	4	1	1	2	1	1
S	3	2	1/2	1	1	1/2
L	2	2	1	1	1	1/2
G	4	3	1	2	2	1

Figure 7

Tier 2 AHP Comparison Matrix with Respect to lc5 (PUA Termination)

	NH	H	E	S	L	G
NH	1	1	1/4	1/5	1/3	1/4
H	1	1	1/4	1/5	1/3	1/4

Tier 2 AHP Comparison Matrix with Respect to lc6 (Products that Pass through PUA Infrastructure)

	NH	H	E	S	L	G
NH	1	1/5	1/7	1/2	1/3	1/4
H	5	1	1/2	1	2	1

E	4	4	1	1	2	1
S	5	5	1	1	2	1
L	3	3	1/2	1/2	1	1/2
G	4	4	1	1	2	1

Figure 8

E	7	2	1	5	3	2
S	2	1	1/5	1	1/2	1/3
L	3	1/2	1/3	2	1	1/2
G	4	1	1/2	3	2	1

Figure 9

Using the appropriate mathematical techniques we extract the necessary eigenvalues and eigenvectors to obtain the weights for the Tier 1 and Tier 2 factors.

Weights		Eigenvalue
infrastructure:	.036	6.301
pre-PUA:	.108	
PUA:	.217	
post-PUA:	.108	
PUA termination:	.050	
products:	.481	

Figure 10. Tier 1 Weights and Eigenvalue

Weighting for the Tier 2 criteria with respect to each of the six Tier 1 criteria is shown in Figure 11.

	infrastructure (lc1)	pre-PUA (lc2)	PUA (lc3)	post-PUA (lc4)	PUA termination (lc5)	products (lc6)
NH	.153	.055	.055	.055	.055	.043
H	.090	.130	.130	.130	.055	.180
E	.298	.211	.211	.211	.244	.364
S	.249	.159	.159	.159	.262	.090
L	.105	.167	.167	.167	.140	.120
G	.105	.279	.279	.279	.244	.202

Figure 11. Tier 2 Weights

For the Tier 2 matrices the eigenvalues range from 6.025 to 6.279. Using Saaty's inconsistency measure (pp. 80-84, Saaty 1995), we see that all of the 6x6 Tier 1 and Tier 2 matrices have acceptable consistency. Combining the Tier 1 and Tier 2 weights yields the cell weighting in the streamlined LCA matrix shown in Figure 12. For example, the weighting factor of .006 in the first cell (lc1, NH) is computed as .036 (weight for lc1 from Figure 10) times 0.153 (weight for NH with respect to lc1 from Figure 11).

	NH	H	E	S	L	G
lc1	.006	.003	.011	.009	.004	.004
lc2	.006	.014	.023	.017	.018	.030
lc3	.012	.028	.046	.034	.036	.060
lc4	.006	.014	.023	.017	.018	.030
lc5	.003	.003	.012	.013	.007	.012
lc6	.021	.087	.175	.043	.058	.097

Figure 12. Cell Weights

At this point there are two choices. First a user can evaluate the processes by scoring the Boeing process matrix and then multiplying the raw scores in each cell by the weighting factors found in Figure 12 – matrix question approach. Alternatively, a user can create a series of AHP comparison matrices for the two processes under consideration - direct comparison approach. Both approaches are shown below.

Matrix Question Approach and Supporting Matrices

The user begins by developing the raw scores based on the answers to a series of questions associated with each cell of the process matrix. Typically a team of process and/or pollution prevention experts would do this ranking. Figures 13 and 14 show the raw scores for the two anodizing processes, as developed during the pilot testing of the process matrix questions.

Chromic acid raw scores

	NH	H	E	S	L	G
lc1	3.0	3.8	1.2	3.6	2.7	2.3
lc2	2.9	2.9	1.7	1.9	2.9	1.8
lc3	1.1	2.3	2.3	2.0	2.9	3.1
lc4	3.0	4.4	1.2	2.9	3.6	3.9
lc5	0.0	4.2	0.0	3.3	3.3	1.2
lc6	0.0	2.8	2.0	1.0	3.0	3.3

Figure 13

Boric/sulfuric acid raw scores

	NH	H	E	S	L	G
lc1	3.0	3.3	1.2	3.6	2.7	1.9
lc2	2.9	2.9	1.7	1.9	2.9	1.8
lc3	1.1	1.2	1.4	1.5	2.5	1.9
lc4	3.0	4.4	1.2	2.9	3.6	3.9
lc5	0.0	3.3	0.0	1.7	1.7	.6
lc6	0.0	1.7	2.0	0.0	1.0	3.3

Figure 14

Multiplying the raw scores with the cell weights as shown in Figure 12 produces the weighted scores for the two processes, shown in Figures 15 and 16. For example, in the case of the chromic acid anodize, the entry 3.0 (raw score) from the lc1/NH cell of Figure 13 is multiplied by the entry .006 (weight) from the lc1/NH cell of Figure 12 to obtain the result .018 (weighted score) found in the lc1/NH cell of Figure 15.

The product of the chromic acid raw scores and the cell weights (Figure 8).

	NH	H	E	S	L	G
lc1	.018	.011	.013	.032	.011	.009
lc2	.017	.041	.039	.032	.052	.054
lc3	.013	.064	.106	.068	.104	.186
lc4	.018	.062	.028	.049	.065	.117
lc5	.000	.013	.000	.043	.023	.014
lc6	.000	.244	.350	.043	.174	.320

Figure 15

The product of the boric/sulfuric acid raw scores and the cell weights (Figure 8).

	NH	H	E	S	L	G
lc1	.018	.010	.013	.032	.011	.008
lc2	.017	.041	.039	.032	.052	.054
lc3	.013	.034	.064	.051	.090	.114
lc4	.018	.062	.028	.049	.065	.117
lc5	.000	.010	.000	.022	.012	.007
lc6	.000	.148	.350	.000	.058	.320

Figure 16

The two processes can now be compared by summing all of the elements in the two respective matrices, by doing row or column sums or even by calculating on a cell-by-cell basis.

Direct Comparison Approach and Supporting Matrices

To carry out a direct comparison of the two process alternatives, the user ranks the two processes against one another with respect to each of the Tier 2 factors for each of the Tier 1 factors. This will result in thirty-six 2x2 matrices. In order to avoid the semantic problems of a double negative (least adverse impact), the user compares alternatives with respect to the level of adverse impact. The user then combines the direct comparisons of the two alternatives with the weighting of the life-cycle stages and environmental concerns to construct a weighting of the two alternatives. From this information, the user then chooses the alternative with the lowest rating as having the least adverse environmental impact. In this case, alternative 1 being more "important" than alternative 2 means that 1 has greater adverse environmental impact than 2. The resulting matrices are shown in Figure 17.

lc1, NH

	A	B
A	1	1
B	1	1

lc1, H

	A	B
A	1	1
B	1	1

lc1, E

	A	B
A	1	1
B	1	1

lc1, S

	A	B
A	1	1
B	1	1

lc1, L

	A	B
A	1	1
B	1	1

lc1, G

	A	B
A	1	1
B	1	1

lc2, NH

	A	B
A	1	1
B	1	1

lc2, H

	A	B
A	1	1
B	1	1

lc2, E

	A	B
A	1	1
B	1	1

lc2, S

	A	B
A	1	1
B	1	1

lc2, L

	A	B
A	1	1
B	1	1

lc2, G

	A	B
A	1	1
B	1	1

lc3, NH

	A	B
A	1	1
B	1	1

lc3, H

	A	B
A	1	5
B	1/5	1

lc3, E

	A	B
A	1	1
B	1	1

lc3, S

	A	B
A	1	7
B	1/7	1

lc3, L

	A	B
A	1	5
B	1/5	1

lc3, G

	A	B
A	1	7
B	1/7	1

lc4 NH

	A	B
A	1	1
B	1	1

lc4, H

	A	B
A	1	1
B	1	1

lc4, E

	A	B
A	1	1
B	1	1

lc4, S

	A	B
A	1	1
B	1	1

lc4, L

	A	B
A	1	1
B	1	1

lc4, G

	A	B
A	1	1
B	1	1

lc5, NH

	A	B
A	1	1
B	1	1

lc5, H

	A	B
A	1	1
B	1	1

lc5, E

	A	B
A	1	1
B	1	1

lc5, S

	A	B
A	1	2
B	1/2	1

lc5, L

	A	B
A	1	2
B	1/2	1

lc5, G

	A	B
A	1	2
B	1/2	1

lc6, NH

	A	B
A	1	1
B	1	1

lc6, H

	A	B
A	1	1
B	1	1

lc6, E

	A	B
A	1	1
B	1	1

lc6, S

	A	B
A	1	4
B	1/4	1

lc6, L

	A	B
A	1	2
B	1/2	1

lc6, G

	A	B
A	1	5
B	1/5	1

Figure 17. Process Alternative AHP Comparison Matrices

As before, the eigenvalues and eigenvectors for these matrices can be computed using available software or by developing appropriate computer algorithms, and from this data, weights can be determined for the alternatives. Conceptually, it may be easier to look at the vector of weights for the alternatives as a score involving the allocation of one point between the alternatives for the corresponding life-cycle stage/area of concern. The user adds the points using the weights that have been determined by the steps discussed in the Cell Valuation Method part of the paper. It is also possible to build up the final "score" for the alternatives by constructing the appropriate matrices and using matrix multiplication to calculate this score. It is important to remember that the scores are relative, and they should not be taken as absolute values associated with each alternative. The matrix construction and the manipulation to accomplish the comparison are detailed below.

The 36 vectors of weights (points) for the alternatives with respect to the 36 life-cycle stage/area of concern combinations are computed as eigenvectors of the matrices of Figure 17 and are shown in Figure 18.

lc1/NH			**lc1/H**			**lc1/E**		
	A:	.500		A:	.500		A:	.500
	B:	.500		B:	.500		B:	.500
lc1/S			**lc1/L**			**lc1/G**		
	A:	.500		A:	.500		A:	.500
	B:	.500		B:	.500		B:	.500
lc2/NH			**lc2/H**			**lc2/E**		
	A:	.500		A:	.500		A:	.500
	B:	.500		B:	.500		B:	.500
lc2/S			**lc2/L**			**lc2/G**		
	A:	.500		A:	.500		A:	.500
	B:	.500		B:	.500		B:	.500
lc3/NH			**lc3/H**			**lc3/E**		
	A:	.500		A:	.833		A:	.500
	B:	.500		B:	.167		B:	.500
lc3/S			**lc3/L**			**lc3/G**		
	A:	.875		A:	.833		A:	.875
	B:	.125		B:	.167		B:	.125
lc4/NH			**lc4/ H**			**lc4/E**		
	A:	.500		A:	.500		A:	.500
	B:	.500		B:	.500		B:	.500
lc4/S			**lc4/L**			**lc4/G**		
	A:	.500		A:	.500		A:	.500
	B:	.500		B:	.500		B:	.500

lc5/NH			**lc5/H**			**lc5/E**	
A:	.500		A:	.500		A:	.500
B:	.500		B:	.500		B:	.500
lc5/S			**lc5/L**			**lc5/G**	
A:	.667		A:	.667		A:	.667
B:	.333		B:	.333		B:	.333
lc6/NH			**lc6/H**			**lc6/E**	
A:	.500		A:	.500		A:	.500
B:	.500		B:	.500		B:	.500
lc6/S			**lc6/L**			**lc6/G**	
A:	.800		A:	.667		A:	.833
B:	.200		B:	.333		B:	.167

Figure 18. Alternative Process Scores

The next step is to take this information and put it together to evaluate the alternatives with respect to each other. Again, it is important to note that these are relative, not absolute, rankings. We do this in two steps. First, we will use the Tier 2 weights and the alternative scores to create a matrix that displays the weighted scores of the two processes with respect to each of the life-cycle stages.
This yields the matrix summaries shown in Figure 19.

Anodizing Process

	lc1	lc2	lc3	lc4	lc5	lc6
A-Chromic	.500	.500	.763	.500	.608	.615
B-Boric/Sulfuric	.500	.500	.237	.500	.392	.385

Figure 19. Process Comparison Combined with Tier 2 Weights

In Figure 19, for example, the factor .763 for process A under column lc3 is computed as (.055)(.5) + (.130)(.833) + (.211)(.5) + (.159)(.875) + (.167)(.833) + (.279)(.875), where the numbers .055, .130, .211, .159, .167, and .279 are taken from Figure 11, and the numbers .5, .833, .5, .875, .833, and .875 are taken from the lc3 portion of Figure 18. From Figure 19, alternative A (chromium acid anodize) is seen to be less environmentally acceptable (higher score) or at best the same as the alternative boric/sulfuric acid anodize for each life-cycle stage. We now can take the weights for the Tier 1 factors with respect to the objective to obtain the final matrix (vector) that compares the alternatives. Doing this yields the values in Figure 20.

A-Chromic:	.618
B-Boric/Sulfuric:	.382

Figure 20. Final Direct Comparison for Alternative Processes

If one considers the direct comparison results of Figure 18 as the allocation of one point between each of the two alternatives for each of the 36 life-cycle/concern combinations, then these scores can be combined with the cell weighting matrix of Figure 12 to create two matrices, one for each process, that are analogs of weighted scores shown in Figures 15 and 16. Doing this results in the matrices shown in Figures 21 and 22.

The product of the chromic acid AHP scores (Figure 18) and the cell weights (Figure 8).

	NH	H	E	S	L	G
lc1	.018	.011	.013	.032	.011	.009
lc2	.017	.041	.039	.032	.052	.054
lc3	.013	.064	.106	.068	.104	.186
lc4	.018	.062	.028	.049	.065	.117
lc5	.000	.013	.000	.043	.023	.014
lc6	.000	.244	.350	.043	.174	.320

Figure 21

The product of the boric/sulfuric acid AHP scores (Figure 18) and the cell weights (Figure 8).

	NH	H	E	S	L	G
lc1	.018	.010	.013	.032	.011	.008
lc2	.017	.041	.039	.032	.052	.054
lc3	.013	.034	.064	.051	.090	.114
lc4	.018	.062	.028	.049	.065	.117
lc5	.000	.010	.000	.022	.012	.007
lc6	.000	.148	.350	.000	.058	.320

Figure 22

If we sum the scores in the matrices of Figures 21 and 22, we obtain (allowing for round-off) the scores of .618 for alternative A and .382 for alternative B as given in the text of the published paper and Figure 20. These results can also be obtained by constructing and combining a series of matrices that utilize the AHP weights and alternative process weights. This is done in the next section.

Part 3 - AHP Analysis Using Matrix Notation

With the advent of user-friendly computational programs such as MatLab and Mathcad, which make use of matrix notation, it is useful to frame the AHP technique as applied to streamlined LCA matrices in the language of matrices.

For this discussion, assume that there are *m* Tier 1 factors, *n* Tier 2 factors for each Tier 1 factor, and *p* alternatives. Additional tiers and/or a variable number of lower tier factors for each factor in the tier above would be handled in a manner similar to the following, but the notation would also become more complicated. The analysis is based on an underlying *mxn* LCA matrix with *m* life-cycle stages and *n* areas of environmental impact. Matrices and vectors (a matrix with only one column (column vector) or one row (row vector)) will be denoted in boldface italics.

Applying the AHP approach to this situation results in one *m*x*m* matrix for the Tier 1 factors, *m n*x*n* matrices for the Tier 2 factors and *mn* (*m* multiplied by *n*) *p*x*p* matrices for the alternatives. The AHP matrices are real, positive, reciprocal symmetric matrices. By this we mean the following:

$\boldsymbol{A} = (a_{ij})$ is any one of the AHP matrices; we have a_{ij} real, $a_{ij} \geq 0$, and:

$$a_{ij} = \begin{cases} 1 & \text{if } i = j \\ 1/a_{ij} & \text{if } i \neq j \end{cases}$$

Under these conditions, Saaty (1996) has demonstrated that for each AHP matrix there is an eigenvalue, denoted λ_{max} of greatest size (modulus), and this eigenvalue is a real, positive number. Corresponding to λ_{max} is an eigenvector whose elements sum to 1, and this eigenvector is the vector of weights (or scores for the alternative level).

We have for the system described above the following vectors:

one (1) Tier 1 vector of weights:	$\boldsymbol{W1}$ with size mx1
m Tier 2 vectors of weights:	$\boldsymbol{W2}_i$ with size nx1, i = 1, ..., m
mn Alternative vectors of weights (scores):	$\boldsymbol{S}_{ij}$ with size px1, i = 1, ..., m, j = 1, ..., n.

These eigenvectors can be obtained from the AHP comparison matrices by using appropriate computer routines. The analyst now uses the augment function to build up matrices from the various eigenvectors. The augment function takes an *r*x*v* matrix, ***V***, and a *r*x*w* matrix, ***W***, and adjoins them to produce a *r*x(*v*+*w*) matrix, ***Z***, with the first v columns of the augmented matrix ***Z*** equal to the columns of ***V*** and the last *w* columns of ***Z*** equal to the columns of ***W***. The relationship is symbolically written as:

$$\boldsymbol{Z} = \text{augment}(\boldsymbol{V},\boldsymbol{W}).$$

The dimensions of the matrices $\boldsymbol{V}$ and $\boldsymbol{W}$ must match correctly in order to define the augment function properly. Using the augment function the analyst can build a series of matrices from the eigenvectors that will enable us to produce the final weighted scores for the p alternatives. The analyst begins with the mn vectors $\boldsymbol{S}_{ij}$ and uses a nested sequence of augment functions to create m pxn matrices $\boldsymbol{AS}_k$, k = 1, ..., m, defined by:

$$\boldsymbol{AS}_{k1} = \text{augment}(\boldsymbol{S}_{k1},\boldsymbol{S}_{k2})$$
$$\boldsymbol{AS}_{k2} = \text{augment}(\boldsymbol{AS}_{k1},\boldsymbol{S}_{k3})$$

.

.

$$\boldsymbol{AS}_k = \text{augment}(\boldsymbol{AS}_{k(n-2)},\boldsymbol{S}_{kn})$$

The $\boldsymbol{AS}_k$ matrices take the scores of the alternatives associated with each of the life-cycle stages, as expressed by the corresponding eigenvectors, and adjoin them to form a matrix with p rows (the number of alternatives) and n columns (the number of areas of impact). The user next computes the m matrix products $\boldsymbol{SL}_i$, i = 1, ..., m defined by:

$$\boldsymbol{SL}_i = \boldsymbol{AS}_i \times \boldsymbol{W2}_i$$

Each $\boldsymbol{SL}_i$ matrix is of size px1. For life-cycle stage j, the first element of $\boldsymbol{SL}_j$ is the score of the first alternative weighted with respect to the areas of impact; the second element is the score of the second alternative; and so on. The user now forms one (1) pxm matrix, $\boldsymbol{FL}$, by augmenting the $\boldsymbol{SL}_i$ matrices as follows:

$$\boldsymbol{FL}_1 = \text{augment}(\boldsymbol{SL}_1, \boldsymbol{SL}_2)$$
$$\boldsymbol{FL}_2 = \text{augment}(\boldsymbol{FL}_1, \boldsymbol{SL}_3)$$

.

.

.

$$\boldsymbol{FL} = \text{augment}(\boldsymbol{FL}_{(m-2)}, \boldsymbol{SL}_m).$$

The columns of $\boldsymbol{FL}$ are the alternative scores, weighted for areas of impact, with column 1 corresponding to life-cycle stage 1, column 2 corresponding to life-cycle stage 2, and so on. The final step is to take the weights associated with the life-cycle steps, $\boldsymbol{W1}$, and combine this data with $\boldsymbol{FL}$ to derive the final scoring, $\boldsymbol{FS}$, for the alternatives. This is accomplished by the following matrix operation:

$$\boldsymbol{FS} = \boldsymbol{FL} \times \boldsymbol{W1}$$

Since $\boldsymbol{FL}$ is a pxm matrix and $\boldsymbol{W1}$ is an mx1 matrix, the result of the multiplication is a px1 matrix. The first element of $\boldsymbol{FS}$ is the score of the first alternative, now weighted for both life-cycle stages and areas of impact; the second element corresponds to the second alternative; and so on. From ***FS*** it is possible to identify the preferred alternative, or alternatives in the event of a tie. If this sequence of matrix operations is carried out for the anodizing process example of the paper, the scores as found in Figure 20 are obtained.