APPENDIX 2

Rovibrational G tensor for penta-atomic amine-like molecules

To illustrate our method, we derive the rovibrational **G** tensor of the simplest amine molecule of a type NH_2OH. It is a prototype of other amines of the first class because more complex molecules can be constructed including additional bonds to this penta-atomic core.

For this molecule the internuclear vectors are chosen according to Figure 1. The determination of the mass coefficients is elementary and gives

$$\left[\frac{1}{m_{i,j}}\right]=\begin{bmatrix} \frac{1}{m_0}+\frac{1}{m_1} & \frac{1}{m_0} & \frac{1}{m_0} & -\frac{1}{m_1} \\ \frac{1}{m_0} & \frac{1}{m_0}+\frac{1}{m_2} & \frac{1}{m_0} & 0 \\ \frac{1}{m_0} & \frac{1}{m_0} & \frac{1}{m_0}+\frac{1}{m_3} & 0 \\ -\frac{1}{m_1} & 0 & 0 & \frac{1}{m_1}+\frac{1}{m_4} \end{bmatrix}.$$

The components of the **G** tensor are simply the dot products of the $\mathbf{s}$ and $\mathbf{\Omega}$ vectors, given by Eqs. (11-13).

The rotation, vibration and Coriolis subtensors of the frame can be considered as the **G** tensor for ammonia-like molecules determined in Ref.(16). As the top part is not included in the construction of the rotating MSA, it contributes only to the vibration and Coriolis tensor elements. The total **G** tensor derived by hands is reported below. It has been checked numerically by the method as described in Ref. (10).

A. 1. The vibrational G tensor of the frame

$$g^{r_1,r_1}=\frac{1}{m_{1,1}},\qquad g^{r_1,r_2}=\frac{\cos^2\alpha+\sin^2\alpha\cos\tau_2}{m_0},\qquad g^{r_1,r_3}=\frac{\cos^2\alpha+\sin^2\alpha\cos\tau_3}{m_0},$$

$$g^{r_2,r_2}=\frac{1}{m_{2,2}},\qquad g^{r_2,r_3}=\frac{\cos^2\alpha+\sin^2\alpha\cos\tau_1}{m_0},\qquad g^{r_3,r_3}=\frac{1}{m_{3,3}},$$

$$g^{r_1,\alpha}=\sin\alpha\cos\alpha\left(\frac{1-\cos\tau_2}{m_0r_2}S_{1,2}+\frac{1-\cos\tau_3}{m_0r_3}S_{1,3}\right),$$

$$g^{r_2,\alpha}=\sin\alpha\cos\alpha\left(\frac{1-\cos\tau_2}{m_0r_1}(S_{1,1}-1)+\frac{1-\cos\tau_1}{m_0r_3}S_{1,3}\right),$$

$$g^{r_3,\alpha}=\sin\alpha\cos\alpha\left(\frac{1-\cos\tau_3}{m_0r_1}(S_{1,1}-1)+\frac{1-\cos\tau_1}{m_0r_3}S_{1,2}\right),$$

$$g^{r_1,\tau_2}=\sin\alpha\cos\alpha\left(\frac{1-\cos\tau_2}{m_0r_2}T_{2,2}+\frac{1-\cos\tau_3}{m_0r_3}T_{2,3}\right)-\frac{\sin\tau_2}{m_0r_2},$$

$$g^{r_1,\tau_3}=\sin\alpha\cos\alpha\left(\frac{1-\cos\tau_2}{m_0r_2}T_{3,2}+\frac{1-\cos\tau_3}{m_0r_3}T_{3,3}\right)-\frac{\sin\tau_3}{m_0r_3},$$

$$g^{r_2,\tau_2}=\sin\alpha\cos\alpha\left(\frac{1-\cos\tau_2}{m_0r_1}T_{2,1}+\frac{1-\cos\tau_1}{m_0r_3}T_{2,3}\right)-\frac{\sin\tau_2}{m_0r_1},$$

$$g^{r_2,\tau_3}=-\sin^2\alpha\left(\frac{\sin\tau_2}{m_0r_1}+\frac{\sin\tau_1}{m_0r_3}\right),$$

$$g^{r_3,\tau_2}=-\sin^2\alpha\left(\frac{\sin\tau_3}{m_0r_1}-\frac{\sin\tau_1}{m_0r_2}\right),$$

$$g^{r_3,\tau_3}=\sin\alpha\cos\alpha\left(\frac{1-\cos\tau_3}{m_0r_1}T_{3,1}+\frac{1-\cos\tau_1}{m_0r_2}T_{3,2}\right)-\frac{\sin\tau_3}{m_0r_1},$$

$$g^{\alpha,\alpha}=\frac{(S_{1,1}-1)^2}{m_{1,1}r_1^2}+\frac{S_{1,2}{}^2}{m_{2,2}r_2^2}+\frac{S_{1,3}{}^2}{m_{3,3}r_3^2}+2\frac{(S_{1,1}-1)S_{1,2}}{m_0r_1r_2}f(\tau_2)+2\frac{(S_{1,1}-1)S_{1,3}}{m_0r_1r_3}f(\tau_3)+2\frac{S_{1,2}S_{1,3}}{m_0r_2r_3}f(\tau_1),$$

$$g^{\alpha,\tau_2}=\frac{(S_{1,1}-1)T_{2,1}}{m_{1,1}r_1^2}+\frac{S_{1,2}T_{2,2}}{m_{2,2}r_2^2}+\frac{S_{1,3}T_{2,3}}{m_{3,3}r_3^2}$$
$$+\frac{(S_{1,1}-1)[\cot\alpha\sin\tau_2+f(\tau_2)T_{2,2}]+S_{1,2}[\cot\alpha\sin\tau_2+f(\tau_2)T_{2,1}]}{m_0r_1r_2}$$
$$+\frac{(S_{1,1}-1)T_{2,3}f(\tau_3)+S_{1,3}T_{2,1}\chi(\tau_3)}{m_0r_1r_3}+\frac{S_{1,2}T_{2,3}f(\tau_1)+S_{1,3}T_{2,2}\chi(\tau_1)}{m_0r_2r_3},$$

$$g^{\alpha,\tau_3}=\frac{(S_{1,1}-1)T_{3,1}}{m_{1,1}r_1^2}+\frac{S_{1,2}T_{3,2}}{m_{2,2}r_2^2}+\frac{S_{1,3}T_{3,3}}{m_{3,3}r_3^2}+\frac{(S_{1,1}-1)T_{3,2}f(\tau_2)+S_{1,2}T_{3,1}\chi(\tau_2)}{m_0r_1r_2}$$
$$+\frac{(S_{1,1}-1)[\cot\alpha\sin\tau_3+f(\tau_3)T_{3,3}]+S_{1,3}[\cot\alpha\sin\tau_3+f(\tau_3)T_{3,1}]}{m_0r_1r_3}$$
$$+\frac{S_{1,3}T_{3,2}f(\tau_1)+S_{1,2}T_{3,3}\chi(\tau_1)}{m_0r_2r_3},$$

$$g^{\tau_2,\tau_2}=\frac{T_{2,1}^2+1+\cot^2\alpha}{m_{1,1}r_1^2}+\frac{T_{2,2}^2+1+\cot^2\alpha}{m_{2,2}r_2^2}+\frac{T_{2,3}^2}{m_{3,3}r_3^2}+2\frac{T_{2,1}T_{2,2}\chi(\tau_2)-\psi(\tau_2)}{m_0r_1r_2}$$
$$+2\frac{T_{2,1}T_{2,3}\chi(\tau_3)}{m_0r_1r_3}+2\frac{T_{2,2}T_{2,3}\chi(\tau_1)}{m_0r_2r_3},$$

$$g^{\tau_3,\tau_3}=\frac{T_{3,1}^2+1+\cot^2\alpha}{m_{1,1}r_1^2}+\frac{T_{3,2}^2}{m_{2,2}r_2^2}+\frac{T_{3,3}^2+1+\cot^2\alpha}{m_{3,3}r_3^2}+2\frac{T_{3,1}T_{3,2}\chi(\tau_2)}{m_0r_1r_2}$$
$$+2\frac{T_{3,1}T_{3,3}\chi(\tau_3)-\psi(\tau_3)}{m_0r_1r_3}+2\frac{T_{3,2}T_{3,3}\chi(\tau_1)}{m_0r_2r_3},$$

$$g^{\tau_2,\tau_3}=\frac{T_{2,1}T_{3,1}+1+\cot^2\alpha}{m_{1,1}r_1^2}+\frac{T_{2,2}T_{3,2}}{m_{2,2}r_2^2}+\frac{T_{2,3}T_{3,3}}{m_{3,3}r_3^2}+\frac{(T_{2,1}T_{3,2}+T_{2,2}T_{3,1})\chi(\tau_2)-\psi(\tau_2)}{m_0r_1r_2}$$
$$+\frac{(T_{2,1}T_{3,3}+T_{2,3}T_{3,1})\chi(\tau_3)-\psi(\tau_3)}{m_0r_1r_3}+\frac{(T_{2,2}T_{3,3}+T_{2,3}T_{3,2})\chi(\tau_1)+\psi(\tau_1)}{m_0r_2r_3}.$$

A. 2. The Coriolis tensor of the frame

$$C_x^{r_1}=\sin\alpha\cos\alpha\left(\frac{1-\cos\tau_2}{m_0r_2}S_{2,2}+\frac{1-\cos\tau_3}{m_0r_3}S_{2,3}\right),$$

$$C_y^{r_1}=-\sin\alpha\cos\alpha\left(\frac{1-\cos\tau_2}{m_0r_2}S_{1,2}+\frac{1-\cos\tau_3}{m_0r_3}S_{1,3}\right),$$

$$C_z^{r_1}=\cot\alpha C_x^{r_1},$$

$$C_x^{r_2} = \sin\alpha\cos\alpha\left(\frac{1-\cos\tau_2}{m_0 r_1}S_{2,1} + \frac{1-\cos\tau_1}{m_0 r_3}S_{2,3}\right),$$

$$C_y^{r_2} = -\sin\alpha\cos\alpha\left(\frac{1-\cos\tau_2}{m_0 r_1}S_{1,1} + \frac{1-\cos\tau_1}{m_0 r_3}S_{1,3}\right),$$

$$C_z^{r_2} = \frac{\sin\tau_2}{m_0 r_1} + \cot\alpha C_x^{r_2},$$

$$C_x^{r_3} = \sin\alpha\cos\alpha\left(\frac{1-\cos\tau_3}{m_0 r_1}S_{2,1} + \frac{1-\cos\tau_1}{m_0 r_2}S_{2,2}\right),$$

$$C_y^{r_3} = -\sin\alpha\cos\alpha\left(\frac{1-\cos\tau_3}{m_0 r_1}S_{1,1} + \frac{1-\cos\tau_1}{m_0 r_2}S_{1,2}\right),$$

$$C_z^{r_3} = \frac{\sin\tau_3}{m_0 r_1} + \cot\alpha C_x^{r_3}.$$

$$\begin{aligned} C_x^{\alpha} &= \frac{(S_{1,1}-1)S_{2,1}}{m_{1,1}r_1^2} + \frac{S_{1,2}S_{2,2}}{m_{2,2}r_2^2} + \frac{S_{1,3}S_{2,3}}{m_{3,3}r_3^2} + \frac{(S_{1,1}-1)S_{2,2} + S_{1,2}S_{2,1}}{m_0 r_1 r_2} f(\tau_2) \\ &+ \frac{(S_{1,1}-1)S_{2,3} + S_{1,3}S_{2,1}}{m_0 r_1 r_3} f(\tau_3) + \frac{S_{1,2}S_{2,3} + S_{1,3}S_{2,2}}{m_0 r_2 r_3} f(\tau_1), \end{aligned}$$

$$\begin{aligned} C_y^{\alpha} = -\Bigg[&\frac{(S_{1,1}-1)S_{1,1}}{m_{1,1}r_1^2} + \frac{{S_{1,2}}^2}{m_{2,2}r_2^2} + \frac{{S_{1,3}}^2}{m_{3,3}r_3^2} + \frac{(S_{1,1}-1)S_{1,2} + S_{1,2}S_{1,1}}{m_0 r_1 r_2} f(\tau_2) \\ &+ \frac{(S_{1,1}-1)S_{1,3} + S_{1,3}S_{1,1}}{m_0 r_1 r_3} f(\tau_3) + \frac{S_{1,2}S_{1,3} + S_{1,3}S_{1,2}}{m_0 r_2 r_3} f(\tau_1) \Bigg], \end{aligned}$$

$$C_z^{\alpha} = \cot\alpha\left(C_x^{\alpha} - \frac{\sin\tau_2}{m_0 r_1 r_2}S_{1,2} - \frac{\sin\tau_3}{m_0 r_1 r_3}S_{1,3}\right),$$

$$\begin{aligned} C_x^{\tau_2} &= \frac{S_{2,1}T_{2,1}}{m_{1,1}r_1^2} + \frac{S_{2,2}T_{2,2}}{m_{2,2}r_2^2} + \frac{S_{2,3}T_{2,3}}{m_{3,3}r_3^2} \\ &+ \frac{S_{2,1}[\cot\alpha\sin\tau_2 + f(\tau_2)T_{2,2}] + S_{2,2}[\cot\alpha\sin\tau_2 + f(\tau_2)T_{2,1}]}{m_0 r_1 r_2} \\ &+ \frac{S_{2,1}T_{2,3} f(\tau_3) + S_{2,3}T_{2,1}\chi(\tau_3)}{m_0 r_1 r_3} + \frac{S_{2,2}T_{2,3} f(\tau_1) + S_{2,3}T_{2,2}\chi(\tau_1)}{m_0 r_2 r_3}, \end{aligned}$$

$$C_y^{\tau_2} = -\left[\frac{S_{1,1}T_{2,1}}{m_{1,1}r_1^2} + \frac{S_{1,2}T_{2,2}}{m_{2,2}r_2^2} + \frac{S_{1,3}T_{2,3}}{m_{3,3}r_3^2} \right.$$
$$+ \frac{S_{1,1}[\cot\alpha \sin\tau_2 + f(\tau_2)T_{2,2}] + S_{1,2}[\cot\alpha \sin\tau_2 + f(\tau_2)T_{2,1}]}{m_0 r_1 r_2}$$
$$\left. + \frac{S_{1,1}T_{2,3} f(\tau_3) + S_{1,3}T_{2,1}\chi(\tau_3)}{m_0 r_1 r_3} + \frac{S_{1,2}T_{2,3} f(\tau_1) + S_{1,3}T_{2,2}\chi(\tau_1)}{m_0 r_2 r_3} \right],$$

$$C_z^{\tau_2} = -\frac{1}{\sin^2 \alpha m_{11} r_1^2} + \frac{(1+\cot^2\alpha)\cos\tau_2}{m_0 r_1 r_2} + \cot\alpha \left(C_x^{\tau_2} - \frac{\sin\tau_2}{m_0 r_1 r_2} T_{2,2} - \frac{\sin\tau_3}{m_0 r_1 r_3} T_{2,3} \right),$$

$$C_x^{\tau_3} = \frac{S_{2,1}T_{3,1}}{m_{1,1}r_1^2} + \frac{S_{2,2}T_{3,2}}{m_{2,2}r_2^2} + \frac{S_{2,3}T_{3,3}}{m_{3,3}r_3^2} + \frac{S_{2,1}T_{3,2} f(\tau_2) + S_{2,2}T_{3,1}\chi(\tau_2)}{m_0 r_1 r_2}$$
$$+ \frac{S_{2,1}[\cot\alpha \sin\tau_3 + f(\tau_3)T_{3,3}] + S_{2,3}[\cot\alpha \sin\tau_3 + f(\tau_3)T_{3,1}]}{m_0 r_1 r_3}$$
$$+ \frac{S_{2,3}T_{3,2} f(\tau_1) + S_{2,2}T_{3,3}\chi(\tau_1)}{m_0 r_2 r_3},$$

$$C_y^{\tau_3} = -\left[\frac{S_{1,1}T_{3,1}}{m_{1,1}r_1^2} + \frac{S_{1,2}T_{3,2}}{m_{2,2}r_2^2} + \frac{S_{1,3}T_{3,3}}{m_{3,3}r_3^2} + \frac{S_{1,1}T_{3,2} f(\tau_2) + S_{1,2}T_{3,1}\chi(\tau_2)}{m_0 r_1 r_2} \right.$$
$$+ \frac{S_{1,1}[\cot\alpha \sin\tau_3 + f(\tau_3)T_{3,3}] + S_{1,3}[\cot\alpha \sin\tau_3 + f(\tau_3)T_{3,1}]}{m_0 r_1 r_3}$$
$$\left. + \frac{S_{1,3}T_{3,2} f(\tau_1) + S_{1,2}T_{3,3}\chi(\tau_1)}{m_0 r_2 r_3} \right],$$

$$C_z^{\tau_3} = -\frac{1}{\sin^2 \alpha m_{1,1} r_1^2} + \frac{(1+\cot^2\alpha)\cos\tau_3}{m_0 r_1 r_3} + \cot\alpha \left(C_x^{\tau_3} - \frac{\sin\tau_2}{m_0 r_1 r_2} T_{3,2} - \frac{\sin\tau_3}{m_0 r_1 r_3} T_{3,3} \right),$$

A. 3. The rotatrional μ tensor

$$\mu_{xx} = \frac{S_{2,1}^2}{m_{1,1}r_1^2} + \frac{S_{2,2}^2}{m_{2,2}r_2^2} + \frac{S_{2,3}^2}{m_{3,3}r_3^2} + 2\frac{S_{2,1}S_{2,2} f(\tau_2)}{m_0 r_1 r_2} + 2\frac{S_{2,1}S_{2,3} f(\tau_3)}{m_0 r_1 r_3} + 2\frac{S_{2,2}S_{2,3} f(\tau_1)}{m_0 r_2 r_3},$$

$$\mu_{yy} = \frac{S_{1,1}^2}{m_{1,1}r_1^2} + \frac{S_{1,2}^2}{m_{2,2}r_2^2} + \frac{S_{1,3}^2}{m_{3,3}r_3^2} + 2\frac{S_{1,1}S_{1,2} f(\tau_2)}{m_0 r_1 r_2} + 2\frac{S_{1,1}S_{1,3} f(\tau_3)}{m_0 r_1 r_3} + 2\frac{S_{1,2}S_{1,3} f(\tau_1)}{m_0 r_2 r_3},$$

$$\mu_{xy} = -\left[\frac{S_{1,1}S_{2,1}}{m_{1,1}r_1^2} + \frac{S_{1,2}S_{2,2}}{m_{2,2}r_2^2} + \frac{S_{1,3}S_{2,3}}{m_{3,3}r_3^2} + \frac{S_{1,2}S_{2,1} + S_{1,1}S_{2,2}}{m_0 r_1 r_2} f(\tau_2) \right.$$
$$\left. + \frac{S_{1,1}S_{2,3} + S_{2,1}S_{1,3}}{m_0 r_1 r_3} f(\tau_3) + \frac{S_{2,2}S_{1,3} + S_{1,2}S_{2,3}}{m_0 r_2 r_3} f(\tau_1) \right],$$

$$\mu_{xz} = \cot\alpha \left(\mu_{xx} - \frac{\sin\tau_2}{m_0 r_1 r_2} S_{2,2} - \frac{\sin\tau_3}{m_0 r_1 r_3} S_{2,3} \right),$$

$$\mu_{yz} = \cot\alpha \left(\mu_{xy} + \frac{\sin\tau_2}{m_0 r_1 r_2} S_{1,2} + \frac{\sin\tau_3}{m_0 r_1 r_3} S_{1,3} \right),$$

$$\mu_{zz} = \frac{1}{\sin^2\alpha m_{1,1} r_1^2} + \cot^2\alpha \left(\mu_{xx} - 2\frac{\sin\tau_2}{m_0 r_1 r_2} S_{2,2} - 2\frac{\sin\tau_3}{m_0 r_1 r_3} S_{2,3} \right).$$

A. 4. The vibrational G tensor involving the top coordinates

$$g^{r_4,r_1} = \frac{\cos\theta}{m_1}, \qquad g^{r_4,r_2} = 0, \qquad g^{r_4,r_3} = 0, \qquad g^{r_4,r_4} = \frac{1}{m_1} + \frac{1}{m_4},$$

$$g^{r_4,\alpha} = -\frac{(S_{1,1} - 1)\sin\theta\cos\phi}{m_1 r_1},$$

$$g^{r_4,\tau_2} = -\frac{(T_{2,1}\sin\alpha\cos\phi - \sin\phi)\sin\theta}{m_1 r_1 \sin\alpha},$$

$$g^{r_4,\tau_3} = -\frac{(T_{3,1}\sin\alpha\cos\phi - \sin\phi)\sin\theta}{m_1 r_1 \sin\alpha},$$

$$g^{\theta,r_1} = -\frac{\sin\theta}{m_1 r_4}, \qquad g^{\theta,r_2} = \frac{\sin\alpha[\cos\alpha\cos\phi(\cos\tau_2 - 1) + \sin\phi\sin\tau_2]}{m_0 r_1},$$
$$g^{\theta,r_4} = -\frac{\sin\theta}{m_1 r_1}, \qquad g^{\theta,r_3} = \frac{\sin\alpha[\cos\alpha\cos\phi(\cos\tau_3 - 1) + \sin\phi\sin\tau_3]}{m_0 r_1},$$

$$g^{\theta,\alpha} = \frac{\cos\phi(S_{1,1}-1)}{m_{1,1}r_1^2} + \frac{S_{1,2}[\cos\phi f(\tau_2) - \cos\alpha\sin\phi\sin\tau_2]}{m_0 r_1 r_2}$$

$$+ \frac{S_{1,3}[\cos\phi f(\tau_3) - \cos\alpha\sin\phi\sin\tau_3]}{m_0 r_1 r_3} - \frac{(S_{1,1}-1)\cos\theta\cos\phi}{m_1 r_1 r_4},$$

$$g^{\theta,\tau_2} = \frac{T_{2,1}\sin\alpha\cos\phi - \sin\phi}{m_{1,1}r_1^2} + \frac{T_{2,2}[\cos\phi f(\tau_2) - \cos\alpha\sin\phi\sin\tau_2]}{m_0 r_1 r_2}$$

$$+ \frac{T_{2,3}[\cos\phi f(\tau_3) - \cos\alpha\sin\phi\sin\tau_3]}{m_0 r_1 r_3} - \frac{\cos\theta(T_{2,1}\sin\alpha\cos\phi - \sin\phi)}{m_1 r_1 r_4 \sin\alpha}$$

$$+ \frac{\cos\alpha\cos\phi\sin\tau_2 + \sin\phi\cos\tau_2}{m_0 r_1 r_2 \sin\alpha},$$

$$g^{\theta,\tau_3} = \frac{T_{3,1}\sin\alpha\cos\phi - \sin\phi}{m_{1,1}r_1^2} + \frac{T_{3,2}[\cos\phi f(\tau_2) - \cos\alpha\sin\phi\sin\tau_2]}{m_0 r_1 r_2}$$

$$+ \frac{T_{3,3}[\cos\phi f(\tau_3) - \cos\alpha\sin\phi\sin\tau_3]}{m_0 r_1 r_3} - \frac{\cos\theta(T_{3,1}\sin\alpha\cos\phi - \sin\phi)}{m_1 r_1 r_4 \sin\alpha},$$

$$+ \frac{\cos\alpha\cos\phi\sin\tau_2 + \sin\phi\cos\tau_3}{m_0 r_1 r_3 \sin\alpha},$$

$$g^{\theta,\theta} = \frac{1}{m_{1,1}r_1^2} + \frac{1}{m_{4,4}r_4^2} - \frac{2\cos\theta}{m_1 r_1 r_4},$$

$$g^{\theta,\phi} = \frac{1}{\sin\alpha}\left[\frac{(S_{2,1}\cos\phi + \cos\alpha\sin\phi)}{m_{1,1}r_1^2} + \frac{S_{2,2}[\cos\phi f(\tau_2) - \cos\alpha\sin\phi\sin\tau_2]}{m_0 r_1 r_2}\right.$$

$$\left. + \frac{S_{2,3}[\cos\phi f(\tau_3) - \cos\alpha\sin\phi\sin\tau_3]}{m_0 r_1 r_3} - \frac{\cos\theta(S_{2,1}\cos\phi + \cos\alpha\sin\phi)}{m_1 r_1 r_4}\right],$$

$$g^{\phi,r_1} = \cos\alpha\left(\frac{1-\cos\tau_2}{m_0 r_2}S_{2,2} + \frac{1-\cos\tau_3}{m_0 r_3}S_{2,3}\right),$$

$$g^{\phi,r_2} = \cos\alpha\left(\frac{1-\cos\tau_2}{m_0 r_2}S_{2,1} + \frac{1-\cos\tau_1}{m_0 r_3}S_{2,3}\right)$$

$$- \frac{\cos\alpha\sin\alpha\cos\theta\sin\phi(1-\cos\tau_2) - (\cos\alpha\sin\theta + \sin\alpha\cos\theta\cos\phi)\sin\tau_2}{m_0 r_1 \sin\theta},$$

$$g^{\phi,r_3} = \cos\alpha\left(\frac{1-\cos\tau_3}{m_0 r_1}S_{2,1} + \frac{1-\cos\tau_1}{m_0 r_2}S_{2,2}\right)$$

$$- \frac{\cos\alpha\sin\alpha\cos\theta\sin\phi(1-\cos\tau_3) - (\cos\alpha\sin\theta + \sin\alpha\cos\theta\cos\phi)\sin\tau_3}{m_0 r_1 \sin\theta},$$

$$g^{\phi,r_4} = -\frac{\sin\theta(S_{2,1}\cos\phi + \cos\alpha\sin\phi)}{m_1 r_1 \sin\alpha},$$

$$
\begin{aligned}
g^{\phi,\alpha} = \frac{1}{\sin\alpha}\Bigg[& \frac{(S_{1,1}-1)S_{2,1}}{m_{1,1}r_1^2} + \frac{S_{1,2}S_{2,2}}{m_{2,2}r_2^2} + \frac{S_{1,3}S_{2,3}}{m_{3,3}r_3^2} + \frac{(S_{1,1}-1)S_{2,2}+S_{1,2}S_{2,1}}{m_0 r_1 r_2} f(\tau_2) \\
& + \frac{(S_{1,1}-1)S_{2,3}+S_{1,3}S_{2,1}}{m_0 r_1 r_3} f(\tau_3) + \frac{S_{1,2}S_{2,3}+S_{1,3}S_{2,2}}{m_0 r_2 r_3} f(\tau_1) \Bigg] \\
- \frac{1}{\sin\theta}\Bigg[& \frac{(S_{1,1}-1)\cos\theta\sin\phi}{m_{1,1}r_1^2} + \frac{S_{1,2}[\cos\theta\sin\phi f(\tau_2) + \cot\alpha\sin\tau_2(\sin\alpha\cos\theta\cos\phi+\cos\alpha\sin\theta)]}{m_0 r_1 r_2} \\
& + \frac{S_{1,3}[\cos\theta\sin\phi f(\tau_3) + \cot\alpha\sin\tau_3(\sin\alpha\cos\theta\cos\phi+\cos\alpha\sin\theta)}{m_0 r_1 r_3} - \frac{(S_{1,1}-1)\sin\phi}{m_1 r_1 r_4} \Bigg],
\end{aligned}
$$

$$
\begin{aligned}
g^{\phi,\tau_2} = & \frac{1}{\sin\alpha}\Bigg[\frac{S_{2,1}T_{2,1}}{m_{1,1}r_1^2} + \frac{S_{2,2}T_{2,2}}{m_{2,2}r_2^2} + \frac{S_{2,3}T_{2,3}}{m_{3,3}r_3^2} + \frac{S_{2,1}T_{2,2}+S_{2,2}T_{2,1}}{m_0 r_1 r_2} f(\tau_2) \\
& + \frac{S_{2,1}T_{2,3}+S_{2,3}T_{2,1}}{m_0 r_1 r_3} f(\tau_3) + \frac{S_{2,2}T_{2,3}+S_{2,3}T_{2,2}}{m_0 r_2 r_3} f(\tau_1) \Bigg] - \frac{\cos\theta\sin\phi}{\sin\theta}\Bigg[\frac{T_{2,1}}{m_{1,1}r_1^2} + \frac{T_{2,2}f(\tau_2)}{m_0 r_1 r_2} + \frac{T_{2,3}f(\tau_3)}{m_0 r_1 r_3} \Bigg] \\
& - \frac{1}{\sin\alpha\sin\theta}\Bigg[\frac{\cos\alpha\sin\tau_2(T_{2,2}(\sin\alpha\cos\theta\cos\phi+\cos\alpha\sin\theta)+\cos\theta\sin\phi)}{m_0 r_1 r_2} \\
& + \frac{T_{2,3}\cos\alpha\sin\tau_3(\sin\alpha\cos\theta\cos\phi+\cos\alpha\sin\theta)}{m_0 r_1 r_3} - \frac{\cos\phi+\sin\alpha\sin\phi T_{2,1}}{m_1 r_1 r_4} \Bigg] \\
& - \frac{S_{2,3}\cos\alpha}{\sin^2\alpha}\Bigg[\frac{\sin\tau_2}{m_0 r_1 r_2} - \frac{\sin\tau_3}{m_0 r_1 r_3} + \frac{\sin\tau_1}{m_0 r_2 r_3} \Bigg] - \frac{\sin\alpha\cos\theta\cos\phi+\cos\alpha\sin\theta}{\sin^2\alpha\sin\theta}\left(\frac{1}{m_{1,1}r_1^2} - \frac{\cos\tau_2}{m_0 r_1 r_2} \right),
\end{aligned}
$$

$$
\begin{aligned}
g^{\phi,\tau_3} = & \frac{1}{\sin\alpha}\Bigg[\frac{S_{2,1}T_{3,1}}{m_{1,1}r_1^2} + \frac{S_{2,2}T_{3,2}}{m_{2,2}r_2^2} + \frac{S_{2,3}T_{3,3}}{m_{3,3}r_3^2} + \frac{S_{2,1}T_{3,2}+S_{2,2}T_{3,1}}{m_0 r_1 r_2} f(\tau_2) \\
& + \frac{S_{2,1}T_{3,3}+S_{2,3}T_{3,1}}{m_0 r_1 r_3} f(\tau_3) + \frac{S_{2,2}T_{3,3}+S_{2,3}T_{3,2}}{m_0 r_2 r_3} f(\tau_1) \Bigg] - \frac{\cos\theta\sin\phi}{\sin\theta}\Bigg[\frac{T_{3,1}}{m_{1,1}r_1^2} + \frac{T_{3,2}f(\tau_2)}{m_0 r_1 r_2} + \frac{T_{3,3}f(\tau_3)}{m_0 r_1 r_3} \Bigg] \\
& - \frac{1}{\sin\alpha\sin\theta}\Bigg[\frac{T_{3,2}\cos\alpha\sin\tau_2(\sin\alpha\cos\theta\cos\phi+\cos\alpha\sin\theta)}{m_0 r_1 r_2} \\
& + \frac{\cos\alpha\sin\tau_3(T_{3,3}(\sin\alpha\cos\theta\cos\phi+\cos\alpha\sin\theta)+\cos\theta\sin\phi)}{m_0 r_1 r_3} - \frac{\cos\phi+\sin\alpha\sin\phi T_{3,1}}{m_1 r_1 r_4} \Bigg] \\
& + \frac{S_{2,2}\cos\alpha}{\sin^2\alpha}\Bigg[\frac{\sin\tau_2}{m_0 r_1 r_2} - \frac{\sin\tau_3}{m_0 r_1 r_3} + \frac{\sin\tau_1}{m_0 r_2 r_3} \Bigg] - \frac{\sin\alpha\cos\theta\cos\phi+\cos\alpha\sin\theta}{\sin^2\alpha\sin\theta}\left(\frac{1}{m_{1,1}r_1^2} - \frac{\cos\tau_3}{m_0 r_1 r_3} \right),
\end{aligned}
$$

$$g^{\phi,\phi} = \frac{1}{\sin^2\alpha}\left(\frac{S_{2,1}^2}{m_{1,1}r_1^2} + \frac{S_{2,2}^2}{m_{2,2}r_2^2} + \frac{S_{2,3}^2}{m_{3,3}r_3^2} + 2\frac{S_{2,1}S_{2,2}f(\tau_2)}{m_0 r_1 r_2} + 2\frac{S_{2,1}S_{2,3}f(\tau_3)}{m_0 r_1 r_3} + 2\frac{S_{2,2}S_{2,3}f(\tau_1)}{m_0 r_2 r_3}\right)$$
$$+\frac{1}{\sin^2\theta m_{4,4}r_4^2} - \frac{2\cot\theta\sin\phi}{\sin\alpha}\left(\frac{S_{2,1}}{m_{1,1}r_1^2} + \frac{S_{2,2}f(\tau_2)}{m_0 r_1 r_2} + \frac{S_{2,3}f(\tau_3)}{m_0 r_1 r_3}\right)$$
$$+\left[\frac{\cot^2\alpha + \cot^2\theta + 2\cot\alpha\cot\theta\cos\phi}{m_{1,1}r_1^2} - 2(\cot\alpha\cot\theta\cos\phi + \cot^2\alpha)\left(\frac{S_{2,2}\sin\tau_2}{m_0 r_1 r_2} + \frac{S_{2,3}\sin\tau_3}{m_0 r_1 r_3}\right)\right]$$
$$+\frac{2}{m_1 r_1 r_4 \sin\theta}\left(\frac{S_{2,1}\sin\phi}{\sin\alpha} - \cot\theta - \cot\alpha\cos\phi\right).$$

A. 5. The Coriolis tensor involving the top coordinates

$$C_x^{r_4} = -\frac{S_{2,1}\sin\theta\cos\phi}{m_1 r_1}, \qquad C_y^{r_4} = \frac{S_{1,1}\sin\theta\cos\phi}{m_1 r_1}, \qquad C_z^{r_4} = -\frac{(S_{2,1}\cos\alpha\cos\phi + \sin\phi)\sin\theta}{m_1 r_1 \sin\alpha},$$

$$C_x^{\theta} = \frac{S_{2,1}\cos\phi}{m_{1,1}r_1^2} - \frac{S_{2,1}\cos\theta\cos\phi}{m_1 r_1 r_4} + \frac{S_{2,2}[\cos\phi f(\tau_2) - \cos\alpha\sin\phi\sin\tau_2]}{m_0 r_1 r_2}$$
$$+\frac{S_{2,3}[\cos\phi f(\tau_3) - \cos\alpha\sin\phi\sin\tau_3]}{m_0 r_1 r_3},$$

$$C_y^{\theta} = -\frac{S_{1,1}\cos\phi}{m_{1,1}r_1^2} + \frac{S_{1,1}\cos\theta\cos\phi}{m_1 r_1 r_4} - \frac{S_{1,2}[\cos\phi f(\tau_2) - \cos\alpha\sin\phi\sin\tau_2]}{m_0 r_1 r_2}$$
$$-\frac{S_{1,3}[\cos\phi f(\tau_3) - \cos\alpha\sin\phi\sin\tau_3]}{m_0 r_1 r_3},$$

$$C_z^{\theta} = \frac{1}{\sin\alpha}\left[\frac{S_{2,1}\cos\alpha\cos\phi + \sin\phi}{m_{1,1}r_1^2} + \frac{S_{2,2}[\cos\phi f(\tau_2) - \cos\alpha\sin\phi\sin\tau_2]\cos\alpha}{m_0 r_1 r_2}\right.$$
$$\left.+\frac{S_{2,3}[\cos\phi f(\tau_3) - \cos\alpha\sin\phi\sin\tau_3]\cos\alpha}{m_0 r_1 r_3} - \frac{(S_{2,1}\cos\alpha\cos\phi + \sin\phi)\cos\theta}{m_1 r_1 r_4}\right],$$

$$C_x^{\phi} = \frac{1}{\sin\alpha}\left(\frac{S_{2,1}^2}{m_{1,1}r_1^2} + \frac{S_{2,2}^2}{m_{2,2}r_2^2} + \frac{S_{2,3}^2}{m_{3,3}r_3^2} + 2\frac{S_{2,1}S_{2,2}f(\tau_2)}{m_0 r_1 r_2} + 2\frac{S_{2,1}S_{2,3}f(\tau_3)}{m_0 r_1 r_3} + 2\frac{S_{2,2}S_{2,3}f(\tau_1)}{m_0 r_2 r_3}\right)$$
$$-\frac{\cos^2\alpha}{\sin\alpha}\left(\frac{S_{2,2}\sin\tau_2}{m_0 r_1 r_2} + \frac{S_{2,3}\sin\tau_3}{m_0 r_1 r_3}\right) + \frac{S_{2,1}\sin\phi}{m_1 r_1 r_4 \sin\theta}$$
$$-\frac{\cos\theta}{\sin\theta}\left(\frac{S_{2,1}\sin\phi}{m_{1,1}r_1^2} + \frac{S_{2,2}[\sin\phi f(\tau_2) + \cos\alpha\cos\phi\sin\tau_2]}{m_0 r_1 r_2} + \frac{S_{2,3}[\sin\phi f(\tau_3) + \cos\alpha\cos\phi\sin\tau_3]}{m_0 r_1 r_3}\right),$$

$$C_y^{\phi} = -\frac{1}{\sin\alpha}\left(\frac{S_{2,1}S_{1,1}}{m_{1,1}r_1^2} + \frac{S_{2,2}S_{1,2}}{m_{2,2}r_2^2} + \frac{S_{2,3}S_{1,3}}{m_{3,3}r_3^2} + \frac{S_{2,1}S_{1,2} + S_{1,1}S_{2,2}}{m_0 r_1 r_2} f(\tau_2) + \frac{S_{1,1}S_{2,3} + S_{2,1}S_{1,3}}{m_0 r_1 r_3} f(\tau_3)\right.$$
$$\left. + \frac{S_{2,2}S_{1,3} + S_{1,2}S_{2,3}}{m_0 r_2 r_3} f(\tau_1)\right) + \frac{\cos^2\alpha}{\sin\alpha}\left(\frac{S_{1,2}\sin\tau_2}{m_0 r_1 r_2} + \frac{S_{1,3}\sin\tau_3}{m_0 r_1 r_3}\right) - \frac{S_{1,1}\sin\phi}{m_1 r_1 r_4 \sin\theta}$$
$$+ \frac{\cos\theta}{\sin\theta}\left(\frac{S_{1,1}\sin\phi}{m_{1,1}r_1^2} + \frac{S_{1,2}[\sin\phi f(\tau_2) + \cos\alpha\cos\phi\sin\tau_2]}{m_0 r_1 r_2} + \frac{S_{1,3}[\sin\phi f(\tau_3) + \cos\alpha\cos\phi\sin\tau_3]}{m_0 r_1 r_3}\right),$$

$$C_z^{\phi} = \frac{\cos\alpha}{\sin^2\alpha}\left(\frac{S_{2,1}^2 + 1}{m_{1,1}r_1^2} + \frac{S_{2,2}^2}{m_{2,2}r_2^2} + \frac{S_{2,3}^2}{m_{3,3}r_3^2} + 2\frac{S_{2,1}S_{2,2} f(\tau_2)}{m_0 r_1 r_2} + 2\frac{S_{2,1}S_{2,3} f(\tau_3)}{m_0 r_1 r_3} + 2\frac{S_{2,2}S_{2,3} f(\tau_1)}{m_0 r_2 r_3}\right)$$
$$- \frac{\cos\alpha(1+\cos^2\alpha)}{\sin^2\alpha}\left(\frac{S_{2,2}\sin\tau_2}{m_0 r_1 r_2} + \frac{S_{2,3}\sin\tau_3}{m_0 r_1 r_3}\right) + \frac{1}{\sin\alpha\sin\theta}\left(\frac{\cos\theta\cos\phi}{m_{1,1}r_1^2} + \frac{S_{2,1}\cos\alpha\sin\phi - \cos\phi}{m_1 r_1 r_4}\right)$$
$$- \frac{\cos\alpha\cos\theta}{\sin\alpha\sin\theta}\left(\frac{S_{2,1}\sin\phi}{m_{1,1}r_1^2} + \frac{S_{2,2}[\sin\phi f(\tau_2) + \cos\alpha\cos\phi\sin\tau_2]}{m_0 r_1 r_2} + \frac{S_{2,3}[\sin\phi f(\tau_3) + \cos\alpha\cos\phi\sin\tau_3]}{m_0 r_1 r_3}\right).$$

Here, the functions $f(\tau_i), \chi(\tau_i), \psi(\tau_i)$ are defined for $i = 1,2,3$ as follows

$$f(\tau_i) = \sin^2\alpha + \cos^2\alpha\cos\tau_i\,,$$

$$\chi(\tau_i) = 1 - \cos\tau_i + f(\tau_i),$$
$$\psi(\tau_i) = \cot^2\alpha + \cos\tau_i.$$