Stability of spherical vesicles in electric fields

Supporting Information

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List of the symbols

β_1 Admittance ratio between the membrane and the exterior solution, defined by eq. (62).

β_3 Admittance ratio between the membrane and the interior solution, defined by eq. (63).

χ Ratio of the interior conductivity to the exterior conductivity, defined in eq. (1).

δ Ratio of the bilayer thickness to the exterior radius of the vesicle, $l_{\mathrm{me}}/r_{\mathrm{ex}}$, defined by eq. (23).

E_0 Magnitude of the electric field.

$\mathscr{E}_{\mathrm{ex}}$ The electric fields in the exterior solution, defined by eq. (8).

$\mathbf{E}_{\mathrm{ex}}$ The time independent part of the electric fields in the exterior solution, defined by eq. (8).

$\mathscr{E}_{\mathrm{in}}$ The electric fields in the interior solution, defined by eq. (8).

$\mathbf{E}_{\mathrm{in}}$ The time independent part of the electric fields in the interior solution, defined by eq. (8) .

$\mathscr{E}_{\mathrm{me}}$ The electric fields in the membrane, defined by eq. (8).

$\mathbf{E}_{\mathrm{me}}$ The time independent part of the electric fields in the membrane, defined by eq. (8).

$\varepsilon_{\mathrm{ex}}$ Dielectric constant of the exterior solution, abbreviated as ε_1.

$\varepsilon_{\mathrm{in}}$ Dielectric constant of the interior solution, abbreviated as ε_3.

$\varepsilon_{\mathrm{me}}$ Dielectric constant of the membrane, abbreviated as ε_2.

ε_w Dielectric constant of water.

η Conductivity and frequency dependent part of the deformation amplitudes.

η_{th} Threshold value of η, defined by eq. (35).

F_{be} Bending energy.

ΔF_{be} The bending energy required to deform a spherical vesicle to ellipsoid.

F Free energy of a vesicle.

κ Bending rigidity.

l_{me} Thickness of bilayer.

M_{sp} Spontaneous curvature.

ν Frequency of the electric field.

ω Angular frequency of the electric field.

$\bar{\omega}$ Rescaled angular frequency defined by eq. (38).

ω_c Frequency of prolate-oblate morphological transition.

ω_s Inverse Maxwell-Wagner charging time.

ω_{th} Transition frequency of the prolate/oblate-sphere transitions.

ϕ Azimuth angle.

ψ_{ex} Admittance of the external solution, abbreviated as ψ_1.

ψ_{in} Admittance of the internal solution, abbreviated as ψ_3.

ψ_{me} Admittance of the membrane, abbreviated as ψ_2.

$Re\beta_1$ Real part of β_1.

$Re\beta_3$ Real part of β_3.

r_{ex} Exterior radius of spherical vesicle.

r_{in} Interior radius of spherical vesicle.

r_{me} Radius of the middle surface of spherical vesicle.

s Amplitude of vesicle deformation.

σ_{ex} Conductivity of the exterior solution, abbreviated as σ_1.

σ_{in} Conductivity of the interior solution, abbreviated as σ_3.

σ_{me} Conductivity of the membrane, abbreviated as σ_2.

$\mathbf{T}$ Maxwell stress applied to a vesicle.

θ Inclination angle.

W_{el} Work done by the Maxwell stresses.

Physical constants

It is instructive to estimate the order of the magnitudes of these parameters from the physical quantities involved in the experiments in sec. 2. The conductivity of the membrane, σ_{me}, is of the order of 10^{-14} S/m,[14] and the conductivities of the exterior/interior solutions, σ_{ex} and σ_{in}, mainly used in the experiments are of the order of 10^{-4} S/m to 10^{-2} S/m. The dielectric constants of the solutions are not very sensitive to the salts. We assume that the dielectric constants of the exterior and the interior solutions, $\varepsilon_{\mathrm{ex}}$ and $\varepsilon_{\mathrm{in}}$, is equal to the dielectric constant ε_w of water, i.e. $\varepsilon_{\mathrm{ex}} = \varepsilon_{\mathrm{in}} = \varepsilon_w$. The dielectric constant $\varepsilon_{\mathrm{me}}$ of the membrane is approximately $2\varepsilon_0$ and the dielectric constant of water is approximately $78\varepsilon_0$, where $\varepsilon_0 = 8.85 \times 10^{-12}$ C/Vm is the dielectric constant of vacuum. β_1 and β_3 are the ratios of the membrane admittance to the admittances of the exterior and interior solutions, respectively, see eqs. (62) and (63).

Bending energy required for prolate/oblate deformation

The bending energy required to deform a vesicle in prolate or oblate shapes is written as eq. (6). This equation is identical to that used by Winterhalter and Helfrich.[1] However, we outline the derivation of eq. (6) to be self-contained.

The bending energy of the vesicle is written as[2–4]

$$F_{\mathrm{be}} = \Delta P \int dV + \lambda \int dS + \frac{1}{2}\kappa \int dS (2M - M_{\mathrm{sp}})^2 + \frac{1}{2}\kappa_G \int dSG. \tag{104}$$

ΔP is the osmotic pressure difference between the exterior and interior solutions and $\int dV$ is the volume of the interior of the vesicle, λ is the tension of the vesicle, M and G are the mean and Gaussian curvatures of the vesicle, κ is the bending rigidity, M_{sp} is the spontaneous curvature, and κ_G is the modulus of Gaussian curvature. According to the Gauss-Bonnet theorem, see e.g. ref.,[5] the bending energies arising from the Gaussian curvature do not change with the elliptic deformation of the spherical vesicle because the topology does not change with the deformation. In the experiments, the osmotic pressure in the exterior is adjusted to be identical to the osmotic pressure in the interior solutions, i.e. $\Delta P = 0$.

A spherical vesicle with small deformation, which is represented by the displacement vector $\mathbf{u}(\theta, \phi)$, is described by

$$\mathbf{r}(\theta, \phi) = r_{\mathrm{me}}\mathbf{e}_r + u_r(\theta)\mathbf{e}_r + u_\theta(\theta)\mathbf{e}_\theta. \tag{105}$$

$\mathbf{e}_r$ and $\mathbf{e}_\theta$ are unit vectors in r- and θ-directions, where the coordinate system is illustrated in Figure 2 in the main text. $u_r(\theta)$ and $u_\theta(\theta)$ are the r- and θ-components of the displacement vector $\mathbf{u}(\theta,\phi)$. $|\mathbf{u}(\theta,\phi)|$ is much smaller than the radius of the vesicle r_{me} in the absence of the deformation, i.e. $|\mathbf{u}(\theta,\phi)| \ll r_{\text{me}}$. In the absence of the electric fields, the stable shape of the vesicle is sphere, i.e. $\mathbf{u}(\theta,\phi)=0$, and r_{me} must be chosen to minimize the bending energy, eq. (104). The bending energy for a spherical vesicle is calculated by substituting eq. (105) into eq. (104) for $\mathbf{u}(\theta,\phi)=0$ as

$$F_{\text{be}} = 4\pi[\lambda r_{\text{me}}^2 + \frac{\kappa}{2}(2+M_{\text{sp}}r_{\text{me}})^2 + \kappa_G]. \tag{106}$$

The minimum of the bending energy is achieved when the tension and the radius of the vesicle satisfy the relationship

$$\lambda = -\frac{\kappa}{r_{\text{me}}}M_{\text{sp}} - \frac{1}{2}\kappa M_{\text{sp}}^2. \tag{107}$$

Because the surface integral of the Gaussian curvature, i.e. the fourth term of eq. (104), is topological invariant, the bending energy associated with the elliptic deformations is due to the mean curvature M and the area element dS. We derive the mean curvature M and the area element by the second order of $\mathbf{u}(\theta,\phi)$ following the prescriptions of differential geometry. The tangents of the surface are calculated by the derivatives of eq. (105) with respect to θ and ϕ as

$$\begin{aligned}
\mathbf{r}_\theta(\theta,\phi) &= r_{\text{me}}\left(1+\frac{u_r}{r_{\text{me}}}+\frac{u'_\theta}{r_{\text{me}}}\right)\mathbf{e}_\theta + r_{\text{me}}\left(\frac{u'_r}{r_{\text{me}}}-\frac{u_\theta}{r_{\text{me}}}\right)\mathbf{e}_r \\
\mathbf{r}_\phi(\theta,\phi) &= r_{\text{me}}\sin\theta\left(1+\frac{u_r}{r_{\text{me}}}+\frac{u_\theta}{r_{\text{me}}\tan\theta}\right)\mathbf{e}_\phi,
\end{aligned} \tag{108}$$

where we omitted the explicit notation of the dependence of u_r and u_θ on θ for simplicity. u'_r and u'_θ are the derivatives of u_r and u_θ, respectively. $\mathbf{e}_\phi$ is the unit vector in ϕ direction, i.e. $\mathbf{e}_\phi \equiv \mathbf{e}_r \times \mathbf{e}_\theta$. The subscript, θ and ϕ, in the left hand side indicate the partial derivative of $\mathbf{r}(\theta,\phi)$ with respect to θ and ϕ. Here, we note that this rule is only applied to the derivative of $\mathbf{r}(\theta,\phi)$

otherwise the subscripts θ and ϕ represent the θ and ϕ components of vectors and tensors. The normal vector $\mathbf{n}(\theta,\phi)$ to the surface of the vesicle is directed parallel to $\mathbf{r}_\theta(\theta,\phi)\times\mathbf{r}_\phi(\theta,\phi)$ and is written as

$$\mathbf{n}(\theta,\phi) \;=\; \left[1-\frac{1}{2}\left(\frac{u_r'}{r_{\mathrm{me}}}-\frac{u_\theta}{r_{\mathrm{me}}}\right)^2\right]\mathbf{e}_r-\left(\frac{u_r'}{r_{\mathrm{me}}}-\frac{u_\theta}{r_{\mathrm{me}}}\right)\left[1-\left(\frac{u_r}{r_{\mathrm{me}}}+\frac{u_\theta'}{r_{\mathrm{me}}}\right)\right]\mathbf{e}_\theta \tag{109}$$

by normalizing $\mathbf{r}_\theta(\theta,\phi)\times\mathbf{r}_\phi(\theta,\phi)$ to a unit vector. The area element dS is calculated as

$$\begin{aligned} dS &\equiv |(\mathbf{r}_\theta(\theta,\phi)d\theta)\times(\mathbf{r}_\phi(\theta,\phi)d\phi)| \\ &= r_{\mathrm{me}}^2\sin\theta d\theta d\phi\left[1+\left(\frac{2u_r}{r_{\mathrm{me}}}+\frac{u_\theta'}{r_{\mathrm{me}}}+\frac{u_\theta}{r_{\mathrm{me}}\tan\theta}\right)\right. \\ &\quad\left.+\left(\frac{u_r}{r_{\mathrm{me}}}+\frac{u_\theta'}{r_{\mathrm{me}}}\right)\left(\frac{u_r}{r_{\mathrm{me}}}+\frac{u_\theta}{r_{\mathrm{me}}\tan\theta}\right)+\frac{1}{2}\left(\frac{u_r'}{r_{\mathrm{me}}}-\frac{u_\theta}{r_{\mathrm{me}}}\right)^2\right]. \end{aligned} \tag{110}$$

The components, $g_{\theta\theta}$, $g_{\theta\phi}$, and $g_{\phi\phi}$, of the first fundamental form are written as

$$\begin{aligned} g_{\theta\theta} &\equiv \mathbf{r}_\theta(\theta,\phi)\cdot\mathbf{r}_\theta(\theta,\phi) \\ &= r_{\mathrm{me}}^2\left[1+2\left(\frac{u_r}{r_{\mathrm{me}}}+\frac{u_\theta'}{r_{\mathrm{me}}}\right)+\left(\frac{u_r}{r_{\mathrm{me}}}+\frac{u_\theta'}{r_{\mathrm{me}}}\right)^2+\left(\frac{u_r'}{r_{\mathrm{me}}}-\frac{u_\theta}{r_{\mathrm{me}}}\right)^2\right] \end{aligned} \tag{111}$$

$$\begin{aligned} g_{\theta\phi} &\equiv \mathbf{r}_\theta(\theta,\phi)\cdot\mathbf{r}_\phi(\theta,\phi) \\ &= 0 \end{aligned} \tag{112}$$

$$\begin{aligned} g_{\phi\phi} &\equiv \mathbf{r}_\phi(\theta,\phi)\cdot\mathbf{r}_\phi(\theta,\phi) \\ &= r_{\mathrm{me}}^2\sin^2\theta\left[1+\frac{u_r}{r_{\mathrm{me}}}+\frac{u_\theta}{r_{\mathrm{me}}\tan\theta}\right]. \end{aligned} \tag{113}$$

The components of the inverse matrix of the fundamental form are

$$
\begin{aligned}
g^{\theta\theta} &\equiv \frac{g_{\phi\phi}}{g} \\
&= \frac{1}{r_{\mathrm{me}}^2}\left[1-2\left(\frac{u_r}{r_{\mathrm{me}}}+\frac{u'_\theta}{r_{\mathrm{me}}}\right)+3\left(\frac{u_r}{r_{\mathrm{me}}}+\frac{u'_\theta}{r_{\mathrm{me}}}\right)^2-\left(\frac{u'_r}{r_{\mathrm{me}}}-\frac{u_\theta}{r_{\mathrm{me}}}\right)^2\right] && (114)\\
g^{\theta\phi} &\equiv -\frac{g_{\theta\phi}}{g} \\
&= 0 && (115)\\
g^{\phi\phi} &\equiv \frac{g_{\theta\theta}}{g} \\
&= \frac{1}{r_{\mathrm{me}}^2\sin^2\theta}\left[1-2\left(\frac{u_r}{r_{\mathrm{me}}}+\frac{u_\theta}{r_{\mathrm{me}}\tan\theta}\right)+3\left(\frac{u_r}{r_{\mathrm{me}}}+\frac{u_\theta}{r_{\mathrm{me}}\tan\theta}\right)^2\right], && (116)
\end{aligned}
$$

where $g \equiv g_{\theta\theta}g_{\phi\phi}-g_{\theta\phi}^2$ is the determinant of the first fundamental form. The components, $L_{\theta\theta}$, $L_{\theta\phi}$, and $L_{\phi\phi}$, of the second fundamental form are

$$
\begin{aligned}
L_{\theta\theta} &\equiv -\mathbf{r}_\theta(\theta,\phi)\cdot\mathbf{n}_\theta(\theta,\phi) \\
&= -r_{\mathrm{me}}\left[1+\frac{u_r}{r_{\mathrm{me}}}+\frac{2u'_\theta}{r_{\mathrm{me}}}-\frac{u''_r}{r_{\mathrm{me}}}-\frac{1}{2}\left(\frac{u'_r}{r_{\mathrm{me}}}-\frac{u_\theta}{r_{\mathrm{me}}}\right)^2\right. \\
&\quad \left.+\left(\frac{u'_r}{r_{\mathrm{me}}}-\frac{u_\theta}{r_{\mathrm{me}}}\right)\left(\frac{2u'_r}{r_{\mathrm{me}}}+\frac{u''_\theta}{r_{\mathrm{me}}}-\frac{u_\theta}{r_{\mathrm{me}}}\right)\right] && (117)\\
L_{\theta\phi} &\equiv -\mathbf{r}_\theta(\theta,\phi)\cdot\mathbf{n}_\phi(\theta,\phi) \\
&= 0 && (118)\\
L_{\phi\phi} &\equiv -\mathbf{r}_\phi(\theta,\phi)\cdot\mathbf{n}_\phi(\theta,\phi) \\
&= -r_{\mathrm{me}}\sin^2\theta\left[1+\frac{u_r}{r_{\mathrm{me}}}+\frac{u_\theta}{r_{\mathrm{me}}\tan\theta}\right] \\
&\quad \times\left[1-\frac{1}{2}\left(\frac{u'_r}{r_{\mathrm{me}}}-\frac{u_\theta}{r_{\mathrm{me}}}\right)^2-\frac{1}{\tan\theta}\left(\frac{u'_r}{r_{\mathrm{me}}}-\frac{u_\theta}{r_{\mathrm{me}}}\right)\left(1-\left(\frac{u_r}{r_{\mathrm{me}}}+\frac{u'_\theta}{r_{\mathrm{me}}}\right)\right)\right]. && (119)
\end{aligned}
$$

The mean curvature M is derived as

$$
\begin{aligned}
2M &\equiv -\nabla\cdot\mathbf{n}(\theta,\phi) \\
&= -\left[g^{\theta\theta}L_{\theta\theta}+g^{\theta\phi}L_{\theta\phi}+g^{\phi\theta}L_{\phi\theta}+g^{\phi\phi}L_{\phi\phi}\right] \\
&= -\frac{2}{r_{\rm me}}\left[1-\frac{1}{2}\left(\frac{2u_r}{r_{\rm me}}+\frac{u_r''}{r_{\rm me}}+\frac{u_r'}{r_{\rm me}\tan\theta}\right)-\left(\frac{u_r'}{r_{\rm me}}-\frac{u_\theta}{r_{\rm me}}\right)^2+\frac{3}{2}\left(\frac{u_r}{r_{\rm me}}+\frac{u_\theta'}{r_{\rm me}}\right)^2\right. \\
&\quad +\frac{1}{2}\left(\frac{u_r}{r_{\rm me}}+\frac{u_\theta}{r_{\rm me}\tan\theta}\right)^2+\frac{1}{2}\left(\frac{u_r'}{r_{\rm me}}-\frac{u_\theta}{r_{\rm me}}\right)\left(\frac{2u_r'}{r_{\rm me}}+\frac{u_\theta''}{r_{\rm me}}-\frac{u_\theta}{r_{\rm me}}\right) \\
&\quad -\left(\frac{u_r}{r_{\rm me}}+\frac{u_\theta'}{r_{\rm me}}\right)\left(\frac{u_r}{r_{\rm me}}+\frac{2u_\theta'}{r_{\rm me}}-\frac{u_r''}{r_{\rm me}}\right) \\
&\quad \left.+\frac{1}{2\tan\theta}\left(\frac{u_r'}{r_{\rm me}}-\frac{u_\theta}{r_{\rm me}}\right)\left(\frac{2u_r}{r_{\rm me}}+\frac{u_\theta'}{r_{\rm me}}+\frac{u_\theta}{r_{\rm me}\tan\theta}\right)\right].
\end{aligned}
\tag{120}
$$

The membrane is incompressible, and the local area must be conserved during the deformation of the vesicle. According to eq. (110), the condition for local area conservation by the first order in $\mathbf{u}(\theta,\phi)/r_{\rm me}$ is written as

$$
\frac{2u_r}{r_{\rm me}}+\frac{u_\theta'}{r_{\rm me}}+\frac{u_\theta}{r_{\rm me}\tan\theta}=0. \tag{121}
$$

The deformation of the vesicle into prolate or oblate is represented by u_r in eq. (4). Eq. (121) suggests that u_θ, which is represented as eq. (5), must be associated with u_r to keep the local area constant. For eqs. (4) and (5), the area element dS and the mean curvature M are calculated as

$$
\begin{aligned}
dS &= r_{\rm me}^2\sin\theta d\theta d\phi\left[1-\frac{1}{4}\left(\frac{s}{r_{\rm me}}\right)^2\sin^4\theta+2\left(\frac{s}{r_{\rm me}}\right)^2\sin^2\theta\cos^2\theta\right] && (122)\\
2M &= -\frac{2}{r_{\rm me}}\left[1+\frac{s}{r_{\rm me}}\left(3\cos^2\theta-1\right)-4\left(\frac{s}{r_{\rm me}}\right)^2\sin^2\theta\cos^2\theta+\frac{5}{4}\left(\frac{s}{r_{\rm me}}\right)^2\sin^4\theta\right]. && (123)
\end{aligned}
$$

By substituting eqs. (122) and (123) into eq. (104) and using eq. (107), the increase of the bending energy arising from the elliptic deformation is derived as

$$
\Delta F_{\rm be} = \frac{48\pi}{5}\left(1-\frac{M_{\rm sp}r_{\rm me}}{6}\right)\kappa\left(\frac{s}{r_{\rm me}}\right)^2. \tag{124}
$$

Quasistatic Maxwell equations and Maxwell-Wagner theory

The electric fields around the vesicle are derived as the solutions of the Maxwell equation. The frequency of the external electric fields is low enough to neglect the generation of induced electric fields by alternating magnetic fields. In this limit, the Maxwell equations are written as[6]

$$\nabla \cdot \mathscr{D}_k(\mathbf{r},t) = \rho_k(\mathbf{r},t) \tag{125}$$

$$\nabla \cdot \mathscr{B}_k(\mathbf{r},t) = 0 \tag{126}$$

$$\nabla \times \mathscr{E}_k(\mathbf{r},t) = 0 \tag{127}$$

$$\nabla \times \mathscr{H}_k(\mathbf{r},t) = \mathbf{j}_k(\mathbf{r},t) \tag{128}$$

for $k = 1$, 2, and 3. The subscripts $k = 1$, 2, and 3 are the abbreviations of ex, me, and in, respectively. ∇ is 3D gradient and $\mathbf{r}$ is the positional vector. $\mathscr{D}_k$ is the electric flux density and ρ_k is the true electric charge density. $\mathscr{B}_k$ is the magnetic flux density. $\mathscr{E}_k$ is the electric field. $\mathscr{H}_k$ is the magnetic field, and $\mathbf{j}_k$ is the electric current density. $\rho_k(\mathbf{r},t)$ is the electric charge density and $\mathbf{j}(\mathbf{r},t)$ is the electric current density.

$\mathscr{D}_k$ is related to $\mathscr{E}_k$ as

$$\mathscr{D}_k(\mathbf{r},t) = \varepsilon_k \mathscr{E}_k(\mathbf{r},t). \tag{129}$$

In general, $\rho(\mathbf{r},t)$ in the right hand side of eq. (125) includes the densities of both the fixed true electric charges and the true electric charges accumulated by the electric current $\mathbf{j}_k$. However, in this paper, we assume that the fixed electric charges are absent, and that $\rho_k(\mathbf{r},t)$ only includes the accumulated true electric charge. $\mathbf{j}(\mathbf{r},t)$ is the electric current density including the Maxwell displacement current. $\mathbf{j}_k(\mathbf{r},t)$ is represented as

$$\mathbf{j}_k(\mathbf{r},t) = \frac{1}{2}\psi_k \mathbf{E}_k(\mathbf{r})\mathrm{e}^{-i\omega t} + c.c., \tag{130}$$

where

$$\psi_k = \sigma_k - i\omega\varepsilon_k \tag{131}$$

is the admittance of the medium. *c.c.* stands for the complex conjugate.

We can solve eqs. (125) - (128) for each dielectric medium one by one with appropriate boundary conditions to derive the electric field. However, because we have neglected the induced electric fields, the electric fields are independent of the magnetic fields in the quasistatic approximation. The continuity equation of the electric current is derived as

$$\nabla \cdot \mathbf{j}_k(\mathbf{r},t) = 0, \tag{132}$$

by taking the divergence to the both sides of eq. (128). The electric field distributions are derived by solving eqs. (127) and (132) simultaneously. Eq. (127) suggests that electric fields are still conservative fields in the quasistatic approximation, and it is possible to define scalar potential $U(\mathbf{r},t)$ such that

$$\mathscr{E}(\mathbf{r},t) = -\nabla U(\mathbf{r},t). \tag{133}$$

Because of eq. (132), the scalar potential $U(\mathbf{r},t)$ satisfies the Laplace equation

$$\nabla^2 U(\mathbf{r},t) = 0. \tag{134}$$

The boundary conditions are derived from eqs. (127) and (132). We consider a rectangular circuit across the interface between the external solution and the membrane shown in Figure 11 (a). Δl_{rec} and w_{rec} are the length of the edges of the rectangular circuit across and along the interface, respectively. In the limit of $\Delta l_{\mathrm{rec}} \to 0$, the line integral of the electric fields along the circuit is calculated as

$$
\begin{aligned}
[\mathscr{E}_{1,\mathrm{t}} - \mathscr{E}_{2,\mathrm{t}}]w_{\mathrm{rec}} &= \lim_{\Delta l_{\mathrm{rec}} \to 0} \oint \mathscr{E}(\mathbf{r},t) \cdot d\mathbf{l} \\
&= \lim_{\Delta l_{\mathrm{rec}} \to 0} \int (\nabla \times \mathscr{E}(\mathbf{r},t)) \cdot d\mathbf{S} \\
&= 0. \qquad (135)
\end{aligned}
$$

$\mathscr{E}_{1,\mathrm{t}}$ and $\mathscr{E}_{2,\mathrm{t}}$ are the tangent components of the electric field at the media 1 and 2 sides of the interface. $d\mathbf{l}$ and $d\mathbf{S}$ are the line element and area element vectors. The area surrounded by the closed circuit is the range of the area integral in the second equation of the right hand side. Eq. (127) is used to derive the last equation of eq. (135). Because eq. (135) must be satisfied for any w_{rec}, the tangent electric fields are continuous across the interface. To derive the other boundary condition, we consider a cylindrical closed surface across the interface between the external solution and the membrane as it is described in Figure 11 (b). S_{cyl} and l_{cyl} are the area of the base and the height of the cylindrical surface. The area integral of the electric current density flowing out of the cylindrical surface is calculated as

$$
\begin{aligned}
(j_{1,\mathrm{n}} - j_{2,\mathrm{n}})S_{\mathrm{cyl}} &= \lim_{\Delta l_{\mathrm{cyl}} \to 0} \oint \mathbf{j}(\mathbf{r},t) \cdot d\mathbf{S} \\
&= \lim_{\Delta l_{\mathrm{cyl}} \to 0} \int dV \nabla \cdot \mathbf{j}(\mathbf{r},t) \\
&= 0. \qquad (136)
\end{aligned}
$$

$j_{1,\mathrm{n}}$ and $j_{2,\mathrm{n}}$ are the normal components of the electric current densities at the 1, i.e. the external solution, and 2, i.e. membrane, sides of the interface. dV is the volume element, and the volume integral expands inside of the cylindrical closed surface. Eq. (132) is used to derive the last equation of eq. (136). Because eq. (136) must be satisfied for any S_{cyl}, the normal components of the electric current density are continuous across the interface. In short, the continuities of the tangent electric fields and the normal electric current densities across the interface provide the boundary conditions to derive the electric fields. The boundary conditions for the interior interface between the membrane and the internal solution are derived by replacing the suffixes 1 and 2 in eqs. (135) and (136) to 2 and 3, respectively, and are the continuities of the tangent electric fields and the normal electric current densities across the interior interface.

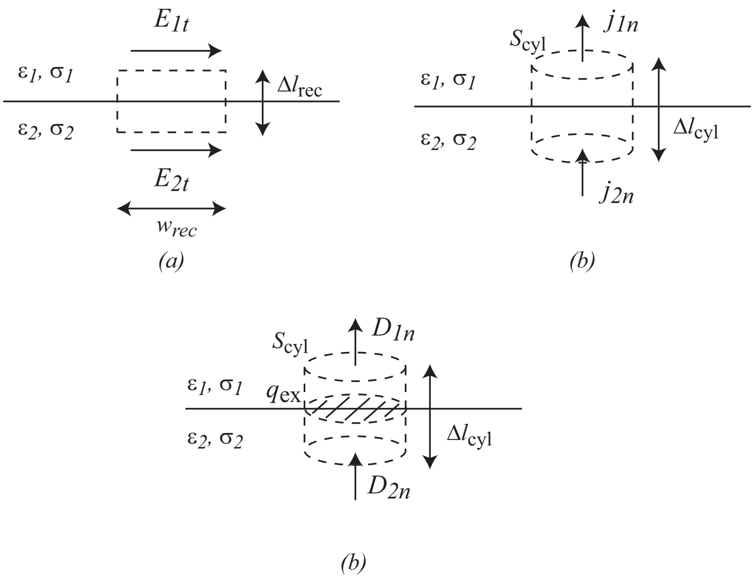

Figure 11: (a) The broken line is the closed circuit, along which the line integral of the first equation of the right hand side of eq. (135) is calculated. $\Delta l_{\rm rec}$ and $w_{\rm rec}$ are the length of the edges of the rectangular circuit across and along the interface, respectively. (b) The cylindrical closed surface, for which the surface integral of the first equation of the right hand side of eq. (135) is calculated. $S_{\rm cyl}$ and $l_{\rm cyl}$ are the area of the base and the height of the cylindrical surface, respectively. (c) The cylindrical closed surface, for which the surface integral of the first equation of the right hand side of eq. (137) is calculated. $S_{\rm cyl}$ and $l_{\rm cyl}$ are the area of the base and the height of the cylindrical surface, respectively. The surface electric charges $q_{\rm ex}(\mathbf{r}, t)$ are accumulated at the interface to make the normal electric current density continuous across the boundary.

In principle, it is possible to derive the electric fields by solving eqs. (127) and (132) simultaneously with the boundary conditions. It is instructive to analyze eq. (125). The argument presented here is the basis for the discussion in sec. 6. In a similar manner to eq. (136), we consider a cylindrical surface with base area $S_{\rm cyl}$ and height $l_{\rm cyl}$ shown in Fig. Figure 11 (c). For $l_{\rm cyl} \to 0$, the area integral of the electric flux density flowing out from the cylindrical surface is calculated as

$$
\begin{aligned}
\mathscr{D}_{1,\mathrm{n}} - \mathscr{D}_{2,\mathrm{n}} &= \lim_{\Delta l_{\mathrm{cyl}} \to 0} \frac{1}{S_{\mathrm{cyl}}} \oint \boldsymbol{\mathscr{D}}(\mathbf{r},t) \cdot d\mathbf{S} \\
&= \lim_{\Delta l_{\mathrm{cyl}} \to 0} \frac{1}{S_{\mathrm{cyl}}} \int dV \nabla \cdot \boldsymbol{\mathscr{D}}(\mathbf{r},t) \\
&= q_{\mathrm{ex}}(\mathbf{r},t), \qquad (137)
\end{aligned}
$$

where $\mathscr{D}_{1,\mathrm{n}}$ and $\mathscr{D}_{2,\mathrm{n}}$ are the normal components of the electric flux densities at 1 and 2 sides of the exterior interface. $q_{\mathrm{ex}}(\mathbf{r},t)$ is the true electric charges accumulated at the interface between the external solution and the membrane. The electric charges are accumulated at the interface in order to make the electric current flow across the interface continuously. A similar relationship is satisfied for the interior interface as

$$
\mathscr{D}_{2,\mathrm{n}} - \mathscr{D}_{3,\mathrm{n}} = q_{\mathrm{in}}(\mathbf{r},t), \qquad (138)
$$

where $\mathscr{D}_{2,\mathrm{n}}$ and $\mathscr{D}_{3,\mathrm{n}}$ are the normal components of the electric flux densities at 2 and 3 sides of the interior interface. $q_{\mathrm{in}}(\mathbf{r},t)$ is the true electric charges accumulated at the interface between the external solution and the membrane. It is important to note that $q_{\mathrm{ex}}(\mathbf{r},t)$ and $q_{\mathrm{in}}(\mathbf{r},t)$ are true electric charges, which are transported by the conduction current, and do not include the electric charges, which are induced by the dielectric polarizations.

Brief Derivation of eqs. (42) and (43)

In this section, we briefly summarize the derivation of eqs. (42) and (43). The reinterpretation of the tangent force densities arising from the shear Maxwell stresses is straight forward. Because the tangent electric fields $\mathscr{E}_{\theta}(r,\theta,t)$ are continuous across an interface, the tangent force densities at the exterior and interior interfaces, see eqs. (19) and (20), are rewritten as

$$f_{\mathrm{ex}\theta}(\theta,t) = \langle q_{\mathrm{ex}}(\theta,t)\mathscr{E}_{\theta}(r_{\mathrm{ex}},\theta,t)\rangle \tag{139}$$

$$f_{\mathrm{in}\theta}(\theta,t) = \langle q_{\mathrm{in}}(\theta,t)\mathscr{E}_{\theta}(r_{\mathrm{in}},\theta,t)\rangle \tag{140}$$

with

$$q_{\mathrm{ex}}(\theta,t) \equiv \varepsilon_{\mathrm{ex}}\mathscr{E}_{\mathrm{ex}r}(r_{\mathrm{ex}},\theta,t) - \varepsilon_{\mathrm{me}}\mathscr{E}_{\mathrm{me}r}(r_{\mathrm{ex}},\theta,t) \tag{141}$$

$$q_{\mathrm{in}}(\theta,t) \equiv \varepsilon_{\mathrm{me}}\mathscr{E}_{\mathrm{me}r}(r_{\mathrm{in}},\theta,t) - \varepsilon_{\mathrm{in}}\mathscr{E}_{\mathrm{in}r}(r_{\mathrm{in}},\theta,t). \tag{142}$$

$\langle\rangle$ represents the time average over one period, $2\pi/\omega$, of the external AC electric field, see sec. 3.4. The quantities $q_{\mathrm{ex}}(\theta,t)$ and $q_{\mathrm{in}}(\theta,t)$ are the area densities of electric charges accumulated at the exterior and interior interfaces, respectively. It is important to note that $q_{\mathrm{ex}}(\theta,t)$ and $q_{\mathrm{in}}(\theta,t)$ represent true electric charges, e.g. ions in the solutions, and the electric charges induced by the dielectric polarizations of the membrane and of the solutions do not give rise to force densities. Eqs. (139) and (140) suggest that the force densities associated with shear Maxwell stresses are, indeed, the force densities arising from the interactions between the electric charges accumulated at the corresponding interface and the tangent electric fields.

The normal force densities associated with the tensile and the compressive contributions to the Maxwell stresses are rewritten in terms of the electric charge densities, $q_{\mathrm{ex}}(\theta,t)$ and $q_{\mathrm{in}}(\theta,t)$, as

$$\begin{aligned} f_{\mathrm{ex}r}(\theta,t) &= \langle q_{\mathrm{ex}}(\theta,t)\bar{\mathscr{E}}_{\mathrm{ex}r}(\theta,t)\rangle - \frac{1}{2}(\varepsilon_{\mathrm{ex}}-\varepsilon_{\mathrm{me}})\langle\mathscr{E}_{\mathrm{ex}r}(r_{\mathrm{ex}},\theta,t)\mathscr{E}_{\mathrm{me}r}(r_{\mathrm{ex}},\theta,t)\rangle \\ &\quad -\frac{1}{2}(\varepsilon_{\mathrm{ex}}-\varepsilon_{\mathrm{me}})\langle\mathscr{E}^2_{\theta}(r_{\mathrm{ex}},\theta,t)\rangle \end{aligned} \tag{143}$$

$$\begin{aligned} f_{\mathrm{in}r}(\theta,t) &= \langle q_{\mathrm{in}}(\theta,t)\bar{\mathscr{E}}_{\mathrm{in}r}(\theta,t)\rangle - \frac{1}{2}(\varepsilon_{\mathrm{me}}-\varepsilon_{\mathrm{in}})\langle\mathscr{E}_{\mathrm{me}r}(r_{\mathrm{in}},\theta,t)\mathscr{E}_{\mathrm{in}r}(r_{\mathrm{in}},\theta,t)\rangle \\ &\quad -\frac{1}{2}(\varepsilon_{\mathrm{me}}-\varepsilon_{\mathrm{in}})\langle\mathscr{E}^2_{\theta}(r_{\mathrm{in}},\theta,t)\rangle \end{aligned} \tag{144}$$

with

$$\bar{\mathscr{E}}_{\mathrm{ex}r}(\theta,t) \equiv \frac{1}{2}(\mathscr{E}_{\mathrm{ex}r}(r_{\mathrm{ex}},\theta,t)+\mathscr{E}_{\mathrm{me}r}(r_{\mathrm{ex}},\theta,t)) \tag{145}$$

$$\bar{\mathscr{E}}_{\mathrm{in}r}(\theta,t) \equiv \frac{1}{2}(\mathscr{E}_{\mathrm{me}r}(r_{\mathrm{in}},\theta,t)+\mathscr{E}_{\mathrm{in}r}(r_{\mathrm{in}},\theta,t)). \tag{146}$$

$\bar{\mathscr{E}}_{\mathrm{ex}r}(\theta,t)$ and $\bar{\mathscr{E}}_{\mathrm{in}r}(\theta,t)$ are the normal electric fields, to which the charges $q_{\mathrm{ex}}(\theta,t)$ and $q_{\mathrm{in}}(\theta,t)$ are exposed, respectively. Thus, the first term of eq. (143) and the first term of eq. (144) represent the force densities arising from the interactions between the electric charges accumulated at the corresponding interface and the normal electric fields. However, at this point, it is not fully obvious whether the normal force densities arising from the tensile and the compressive contributions to the Maxwell stresses are reinterpreted in terms of the electric charge densities accumulated at the interfaces because of the additional terms in eqs. (143) and (144).

Instead of treating the force densities applied to the exterior and the interior interfaces individually, here, we consider the net force densities applied at a unit area of the membrane, which is written as

$$\mathbf{f}_{\mathrm{me}}(\theta) = \mathbf{f}_{\mathrm{ex}}(\theta) + \mathbf{f}_{\mathrm{in}}(\theta)(1-\delta)^2. \tag{147}$$

The area of the interior interface is smaller than the area of the exterior interface because of the finite thickness of the bilayer membrane. The factor $(1-\delta)^2$ in the second term of eq. (41) is the correction for the area difference, see also eq. (21).

The force density $\mathbf{f}_{\mathrm{me}}(\theta)$ includes high order terms with respect to β_1, β_3, and δ, which are orders of magnitudes smaller than 1. We calculate eq. (147) while neglecting those terms. The force densities arising from the Maxwell stresses applied from the membrane, $\langle \mathbf{T}_2(r_{\mathrm{ex}},\theta,t)\rangle$ and $\langle \mathbf{T}_2(r_{\mathrm{in}},\theta,t)\rangle$, are smaller than the force densities arising from the Maxwell stresses applied from the solutions, $\langle \mathbf{T}_1(r_{\mathrm{ex}},\theta,t)\rangle$ and $\langle \mathbf{T}_3(r_{\mathrm{in}},\theta,t)\rangle$, at least, by a factor of $\varepsilon_{\mathrm{me}}/\varepsilon_w$. More precisely, the terms, which are lower order than the force densities arising from $\langle \mathbf{T}_1(r_{\mathrm{ex}},\theta,t)\rangle$ and $\langle \mathbf{T}_3(r_{\mathrm{in}},\theta,t)\rangle$, in the force densities arising from $\langle \mathbf{T}_2(r_{\mathrm{ex}},\theta,t)\rangle$ and $\langle \mathbf{T}_2(r_{\mathrm{in}},\theta,t)\rangle$ exactly

cancel each other in the calculation of the work $W_{\rm el}$. This cancellation is independent of the symmetry of the conductivity conditions across the membrane and of the frequency regimes; the same same cancellation was previously found in ref.[1] We omit the force densities arising from $\langle \mathbf{T}_2(r_{\rm ex},\theta,t)\rangle$ and $\langle \mathbf{T}_2(r_{\rm in},\theta,t)\rangle$ assuming $\varepsilon_{\rm me}/\varepsilon_w \ll 1$. In addition, we have assumed that the dielectric constants of the solutions are not sensitive to the salt concentrations, which implies $\varepsilon_{\rm ex}=\varepsilon_{\rm in}=\varepsilon_w$ and, thus, $(\varepsilon_{\rm ex}-\varepsilon_{\rm in})\langle \mathscr{E}_{{\rm ex}r}\mathscr{E}_{{\rm in}r}\rangle = 0$ in the r-component of $\mathbf{f}_{\rm me}(\theta)$. We further neglect the term, $\varepsilon_w\langle \mathscr{E}_{{\rm ex}r}(\mathscr{E}_\theta(r_{\rm ex})-\mathscr{E}_\theta(r_{\rm in}))\rangle$, in the θ-component of $\mathbf{f}_{\rm me}(\theta)$ since it represents only a correction term. Indeed, this term leads to the $Re\beta_1/\delta$ term in eq. (24) and to the $\delta Re\beta_1$ term in eq. (26), both of which were neglected in deriving the asymptotic expressions for the work $W_{\rm el}$ in sec. 4.

Finally, the components of the net force densities, $\mathbf{f}_{\rm me}=(f_{{\rm me}r}(\theta), f_{{\rm me}\theta}(\theta), f_{{\rm me}\phi}(\theta))$ in spherical coordinate system, are represented as

$$f_{{\rm me}r}(\theta) = \langle q_{\rm me}(\theta,t)\bar{\mathscr{E}}_r(\theta,t)\rangle - \frac{1}{2}\varepsilon_w\langle \mathscr{E}_\theta^2(r_{\rm ex},\theta,t)-\mathscr{E}_\theta^2(r_{\rm in},\theta,t)\rangle \tag{148}$$

$$f_{{\rm me}\theta}(\theta) = \langle q_{\rm me}(\theta,t)\mathscr{E}_\theta(r_{\rm in},\theta,t)\rangle. \tag{149}$$

with

$$\bar{\mathscr{E}}_r(\theta,t) \equiv \frac{1}{2}(\mathscr{E}_{{\rm ex}r}(r_{\rm ex},\theta,t)+\mathscr{E}_{{\rm in}r}(r_{\rm in},\theta,t)) \tag{150}$$

$$q_{\rm me}(\theta,t) \equiv \varepsilon_{\rm ex}\mathscr{E}_{{\rm ex}r}(r_{\rm ex},\theta,t)-\varepsilon_{\rm in}\mathscr{E}_{{\rm in}r}(r_{\rm in},\theta,t), \tag{151}$$

and $f_{{\rm me}\phi}(\theta)=0$. Eqs. (42) and (43) recover the asymptotic expressions of the work $W_{\rm el}$, eqs. (24), (26), and (33), for all frequency regimes except for the correction terms $Re\beta_1/\delta$ in eq. (24) and $\delta Re\beta_1$ in eq. (26) that are smaller than the other terms of leading order.

References

(1) M. Winterhalter and W. Helfrich, J. Coll. Interf. Sci., **122**, 583 (1988).

(2) P. B. Canham, J. Theor. Biol. **26**, 61 (1970).

(3) W. Helfrich, Z. Naturforsch. **28c**, 693 (1973).

(4) E. A. Evans, Biophys. J. **14**, 923 (1974).

(5) C. E. Weatherburn, *Differential Geometry in Three Dimensions*, Cambridge University Press, Cambridge, 1972.

(6) J.D.Jackson, *Classical Electrodynamics*, John Wiley and Sons, Inc., 1962.