Supporting information

Exact Analytical Solutions of Counter-Current Imbibition with both Capillary and Gravity Effects

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Appendix A

Referring to Equation (1) in the text as fundamental equation of continuity, for two-phase immiscible, incompressible, isotherm flow through a rigid 3D porous medium, the fluid phase α satisfies $P = \rho_\alpha \vec{q}_\alpha$ and Equation (1) becomes:

$$\varphi \frac{\partial(\rho_\alpha S_\alpha)}{\partial t} = -\nabla.(\rho_\alpha \vec{q}_\alpha), \qquad S_n + S_w = 1 \tag{A-1}$$

S_α is the phase saturation, and q_α is the phase velocity which is given by the extended Darcy equation [45]:

$$\vec{q}_\alpha = -\lambda_\alpha(S_\alpha, C)\vec{K}(\nabla p_\alpha(S_\alpha, C) - \rho_\alpha \vec{g}) \tag{A-2}$$

Mobility $\lambda_\alpha = k_{r\alpha}(S_\alpha, C)/\mu_\alpha(C)$ is the ratio of relative permeability $k_{r\alpha}$ of phase α to its viscosity, μ_α. Mobility ratio describes the impairment of the flow of one phase by the other phase. K is the permeability tensor of the porous medium, p_α is the fluid pressure of phase α (which is defined through capillary pressure as $p_c = p_n - p_w$) and g is the gravitational acceleration vector.

It should be noted that Equations (1) and (A-1) can be derived from first principles [45], while equation (A-2) is an assumption.

The parameter C in Equation (A-2) represents chemical composition (water salinity, polymer, additives, tracers, etc.), with the following assumptions:

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i. Components do not alter the porous medium (through chemical reactions);
ii. They do not partition into the other phase;
iii. Solute mass flux due to hydrodynamic dispersion within a phase is described by a Fickian model;
iv. The volume fraction of the chemical composition is small compared to that of the wetting phase;
v. The only chemical interaction between the rock and the components is equilibrium adsorption [46].

The flux of any component C_i includes contributions from advection and dispersion [47]:

$$\Im_{C_i} = C_i F_w - \varphi S_w \overline{\overline{D}} \nabla C_i \tag{A-3}$$

where D is the hydrodynamic dispersion tensor [47]:

$$\overline{\overline{D}} = \alpha_T \|\overline{\overline{q}}_w\| \delta_{i,j} + (\alpha_L - \alpha_T) \overline{\overline{q}}_{w,i} \overline{\overline{q}}_{w,j} / \|\overline{\overline{q}}_w\| \tag{A-4}$$

$\delta_{i,j}$ is the Kronecker delta, α_L and α_T are longitudinal and transversal dispersivity, respectively, which are measured through experiments. D accounts for the effects of velocity fields, but the effect of molecular diffusion is ignored.

For a system of n components, there are n advection-dispersion-reaction (ADR) equations:

$$\phi \frac{\partial}{\partial t}(\rho_w C_i S_w) + \nabla \cdot (\rho_w C_i q_w) = \nabla \cdot \left(\phi S_w \rho_w \overline{\overline{D}} \nabla C_i\right) - \frac{\partial \left[(1-\phi)\rho_r A_{s,i} \rho_w\right]}{\partial t} \tag{A-5}$$

The rock density is ρ_r, and the adsorption of C_i per unit mass of rock is presented by $A_{s,i}$. In addition, when water fluid properties stay constant, the tracer concentration will be constant (e.g., $\rho_w(C) = \rho_w = const.$ and $\mu_w(C) = \mu_w = const.$). For chemical components (polymer or salts), empirical adsorption relations can be used [48] as below:

$$\Gamma := \frac{(1-\varphi)}{\varphi} \rho_r A_s \tag{A-6}$$

At thermodynamic equilibrium, isotherm is a function of C only (i.e. $\Gamma = \Gamma(C)$). Furthermore, variations in density (due to the presence of chemical components) are considered in the gravity term only.

Equation (A-2) combined with the definition of capillary pressure results in a parabolic Partial differential Equation for water pressure [49]:

$$\varphi c_t \frac{\partial p_w}{\partial t} = \nabla.\left(\vec{K}\lambda_t(S_w,C)\nabla p_w - \vec{K}\lambda_n(S_w,C)\nabla p_c - (\rho_w\lambda_w(S_w,C)+\rho_n\lambda_n(S_w,C))\vec{g}\right) \quad \text{(A-7)}$$

where c_t is the total compressibility of the fluid-rock system, $\lambda_t = \lambda_w + \lambda_n$, and $\lambda_\alpha = k_{r\alpha}(S_\alpha, C)/\mu_\alpha(C), \alpha \in \{w,o\}$.

The fractional flow function is also introduced as:

$$f_w(S_w,C) = \left(1 + \frac{k_{rn}(S_w)\mu_n}{k_{rw}(S_w,C)\mu_w(C)}\right)^{-1} = \left(\frac{\frac{k_{rw}}{\mu_w}}{\frac{k_{rw}}{\mu_w}+\frac{k_{rn}}{\mu_n}}\right) \quad \text{(A-8)}$$

$$f_w(S_w,C) = \frac{\lambda_w}{\lambda_w+\lambda_n} = \frac{\lambda_w}{\lambda_t}$$

The wetting and non-wetting relative permeability functions depend on saturation (S_w) only, and numerous correlations have been proposed for them, already.

The total velocity is $\vec{q}_t = \vec{q}_w + \vec{q}_n$. The set of equations is obtained as:

$$\begin{cases} \vec{q}_w = f\vec{q}_t - \vec{K}\frac{fk_{ro}}{\mu_n}\nabla p_c + \frac{\lambda_w\lambda_n}{\lambda_t}(\rho_w-\rho_n)\vec{g} \\ \phi\frac{\partial S_w}{\partial t} = -\nabla\cdot(f\vec{q}_t) + \nabla\cdot\left(\vec{K}\frac{fk_{ro}}{\mu_n}\nabla p_c\right) - \nabla\cdot\left(\frac{\lambda_w\lambda_n}{\lambda_t}(\rho_w-\rho_n)\vec{g}\right) + \frac{P}{\rho_w} \\ \phi\frac{\partial(CS_w)}{\partial t} + \frac{\partial\Gamma}{\partial t} = -\nabla\cdot(Cf(S_w,C)\vec{q}_t) - C\vec{K}\frac{fk_{ro}}{\mu_n}(S_w,C)\nabla p_c \\ +C\frac{\lambda_w\lambda_n}{\lambda_t}(S_w,C)\vec{K}\nabla\cdot((\rho_w(C)-\rho_n)\vec{g}) + \nabla\cdot(\phi S_w\vec{D}\nabla C) \end{cases} \quad \text{(A-9)}$$

When making the additional assumptions below for previously suggested models:

i. No dissolved components exist, C = 0;
ii. The porous medium is homogeneous;
iii. The medium is one-dimensional and horizontal so that gravity can be ignored;
iv. There is no sink or source term.

Applying these assumptions, the final equation form would be like below:

$$\varphi \frac{\partial S_w}{\partial t} = -q_t \frac{df}{dS_w}\frac{\partial S_w}{\partial x} + \frac{\partial}{\partial x}\left(D(S_w)\frac{dp_c}{dS_w}\frac{\partial S_w}{\partial x} \right) \quad \text{(A-10)}$$

$D(S_w)$ has also the following equation form:

$$D(S_w) = -K\frac{\lambda_w \cdot \lambda_o}{\lambda_t}\frac{dP_c}{dS_w} \quad \text{(A-11)}$$

Appendix B

The net accumulated volume of wetting fluid is:

$$V_w(t) = \int_0^t (q_0(\tau) - f(S_{wi})q_t)d\tau = \int_0^t q_0(\tau)(1 - f(S_{wi})R)d\tau = 2A(1 - f(S_{wi})R)t^{+1/2} \quad \text{(B-1)}$$

where R is the ratio of total flow to water flux at x = 0 ($R = q_t/q_0$). Clearly, for unidirectional displacement, $q_t = q_0$ and R = 1. For counter-current flow, total flux is equal to zero and R = 0.

The total imbibed volume of water would be:

$$Q_w(t) = \int_0^t q_w(x = 0, t)\,dt = 2A\sqrt{t} \quad \text{(B-2)}$$

The magnitude of A determines the speed of SI, and it is a function of the rock-fluid system. The fractional flow functions for spontaneous imbibition are [1]:

$$F(x,t) = \frac{q_w/q_0 - f_i R}{1 - f_i R} \quad \text{(B-3)}$$

With R = 0 for counter-current imbibition and Equation (B-3) and Equation (11) turns into:

$$F(x,t) = \frac{q_w}{q_0} \Rightarrow q_w = F \cdot q_0 = \frac{FA}{\sqrt{t}} \quad \text{(B-4)}$$

When considering Equation (A-2), the general 1D continuity equation simplifies to:

$$\varphi \frac{\partial S_w}{\partial t} = -\frac{\partial q_w}{\partial x} \quad \text{(B-5)}$$

Substituting Equation (B-4) into Equation (B-5), and using the similarity variable $\omega = x/\sqrt{t}$:

$$\varphi\frac{\partial S_w}{\partial\omega}\frac{\partial\omega}{\partial t}=-\left(\frac{A}{\sqrt{t}}\right)\frac{\partial F}{\partial S_w}\frac{\partial S_w}{\partial\omega}\frac{\partial\omega}{\partial x}\Rightarrow\varphi\frac{dS_w}{d\omega}\left(-\frac{\omega}{2t}\right)=-\left(\frac{A}{\sqrt{t}}\right)\frac{dF}{dS_w}\frac{dS_w}{d\omega}\left(\frac{1}{\sqrt{t}}\right)$$
$$\Rightarrow\omega=\frac{2A}{\varphi}\frac{dF}{dS_w} \qquad \text{(B-6)}$$

The derivative of Equation (B-6) is:

$$\frac{d\omega}{dS_w}=\frac{2A}{\varphi}\frac{d^2F}{dS_w^2} \qquad \text{(B-7)}$$

Now consider Equation (A-9) for q_w (where $q_t = 0$):

$$q_w=fq_t-D(S_w)\frac{\partial S_w}{\partial x}\xrightarrow{q_t=0}q_w=-D(S_w)\frac{\partial S_w}{\partial x} \qquad \text{(B-8)}$$

Combined with Equation (B-4):

$$q_w=\frac{F\cdot A}{\sqrt{t}}=-D(S_w)\frac{\partial S_w}{\partial\omega}\frac{\partial\omega}{\partial x}\Rightarrow\frac{F\cdot A}{\sqrt{t}}=-D(S_w)\frac{dS_w}{d\omega}\left(\frac{1}{\sqrt{t}}\right)$$
$$\Rightarrow\frac{dS_w}{d\omega}=-\frac{F\cdot A}{D(S_w)}\quad\rightarrow\quad\frac{d\omega}{dS_w}=-\frac{D(S_w)}{F\cdot A} \qquad \text{(B-9)}$$

Inserting Equation (B-9) in Equation (B-7):

$$-\frac{D(S_w)}{F\cdot A}=\frac{2A}{\phi}\frac{d^2F}{dS_w^2}\Rightarrow\frac{d^2F}{dS_w^2}=-\frac{\phi}{2A^2}\frac{D(S_w)}{F} \qquad \text{(B-10)}$$

This is an Ordinary Differential Equation (ODE) with respect to F. The solution for Equation (B-10) can be obtained by integrating twice and applying initial and boundary conditions:

$$F(S_w)=1-\frac{\int_{S_w}^{S_0}\frac{(\beta-S_w)D}{F}d\beta}{\int_{S_i}^{S_0}\frac{(S_w-S_i)D}{F}dS_w} \qquad \text{(B-11)}$$

Appendix C

Since F also depends on the constant A, iteration for A is required; in this case, an initial estimation of A is required. As $F=1$ at the inlet, iteration of A is repeated until $F(x=0,t)=1$. The backward derivative of F" is as follows:

$$F''(S_w) \approx \frac{F(S_w + 2\Delta S_w) - 2F(S_w + \Delta S_w) + F(S_w)}{\Delta S_w^2} \quad \text{(C-1)}$$

Considering Equation (B-9) and rearranging for counter-current imbibition, we have:

$$F(S_w) = \left[F(S_w + \Delta S_w) - 0.5F(S_w + 2\Delta S_w)\right] + \sqrt{\left[F(S_w + \Delta S_w) - 0.5F(S_w + 2\Delta S_w)\right]^2 - \left(\frac{\varphi}{2A^2}\right) D(S_w) \Delta S_w^2} \quad \text{(C-2)}$$

When using the numerical form as in Equation (C-2), we can find an approximation of $F(S_w)$ knowing $F(S_w + \Delta S_w)$ and $F(S_w + 2\Delta S_w)$ beforehand, based on some starting value of $F(S_{w,Start})$. The value of $F(x = 0, t) = 1$ is considered as that starting value. The other required value of $F(S_{w,\max} - \Delta S_w)$ is obtained by expansion into a Taylor series:

$$F(S_{w,\max} - \Delta S_w) = F(S_{w,\max}) - \Delta S_w \cdot F'(S_{w,\max}) = 1 - 0 = 1 \quad \text{(C-3)}$$

It is assumed that the maximum saturation is always provided at the inlet and consequently $F'(S_{w,\max}) = 0$.

Convergence of F is achieved by applying two criteria, firstly, as F is a fractional flow function and we have no flow at initial water saturation (S_{wir}), then $F(S_{wir}) = 0$; secondly, due to the mass balance condition, the integrated saturation curve must equal the total imbibed pore volume, i.e.:

$$\int_{S_{wir}}^{S_{w,\max}} x(S_w, t)\, dS_w = \frac{Q_w(t)}{\varphi} = \frac{2A\sqrt{t}}{\varphi} \quad \text{(C-4)}$$

Again, a numerical approximation is applied for Equation (C-4) to give:

$$\sum_{i=1}^{n} F'(S_w, i) \cdot \Delta S_w \approx \frac{Q_w(t)}{\varphi} = \frac{\varphi}{2A\sqrt{t}} = 1 \quad \text{(C-5)}$$

References

45. Hubbert, M.K., *Physical Principles of Oil Production*. 1950, JSTOR.
46. Pope, G.A., *The Application of Fractional Flow Theory to Enhanced Oil Recovery.* Society of Petroleum Engineers Journal, 1980. **20**(03): p. 191-205.
47. Bear, J., *Dynamics of fluids in porous media*. 2013: Courier Corporation.

48. Herbert, A.W., C.P. Jackson, and D.A. Lever, *Coupled groundwater flow and solute transport with fluid density strongly dependent upon concentration.* Water Resources Research, 1988. **24**(10): p. 1781-1795.
49. Peaceman, D.W., *Fundamentals of numerical reservoir simulation*. 2000: Elsevier.