A New Approach to Design Alarm Filters Using the Plant and Controller Knowledge

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1 Proof of Lemma 1

Proof. From the alarm signal formulation (see (8)) we obtain

$$w(z^{-1}) = \lambda_y(z^{-1})y(z^{-1}) + \lambda_u(z^{-1})u(z^{-1}),$$

where $\lambda_y(z^{-1}) = \lambda_{y_1} + \lambda_{y_2}z^{-1} + \cdots + \lambda_{y_r}z^{-r+1}$ and $\lambda_u(z^{-1}) = \lambda_{u_1} + \lambda_{u_2}z^{-1} + \cdots + \lambda_{u_r}z^{-r+1}$. Substituting (10) and (11) in the above equation yields

$$w(z^{-1}) = \frac{\lambda_y(z^{-1})f(z^{-1}) + \lambda_u(z^{-1})g(z^{-1})}{\big(\alpha(z^{-1})f(z^{-1}) - \beta(z^{-1})g(z^{-1})\big)z^{-\zeta}}\nu(z^{-1}), \tag{A.1}$$

where $z^{-\zeta}$ is added in the denominator to make the transfer function proper. The numerator degree of transfer function in (A.1) is $r+l-2$ and the denominator degree is $l+\max(p,q)-2$. So we set $\zeta = r - \max(p,q)$ to have a proper transfer function. For derivation of (A.1), the terms that are associated with b are excluded as they are constant in the steady-state and do not affect the variance. This transfer function can be represented in the state-space form where the system matrices are

$$\mathcal{A} = \left[\begin{array}{c|c} \mathbf{0}^T & I \\ \hline \multicolumn{2}{c}{[\mathbf{0}_\zeta \;\; \boldsymbol{\beta} * \boldsymbol{g} - \boldsymbol{\alpha} * \boldsymbol{f}]} \end{array}\right],$$
$$\mathcal{B} = \begin{bmatrix}\mathbf{0} & 1\end{bmatrix}^T,$$
$$\mathcal{C} = \boldsymbol{\lambda_y} * \boldsymbol{f} + \boldsymbol{\lambda_u} * \boldsymbol{g},$$
$$\mathcal{D} = 0.$$

According to Ref. 1, the state covariance of this system can be found by solving the following Lyapunov equation

$$\mathcal{A}\Psi\mathcal{A}^T - \Psi + \mathcal{B}\mathcal{B}^T = 0. \tag{A.2}$$

As we have assumed that the controller is stabilizing, a solution for (A.2) can be found as

$$\Psi = \sum_{i=0}^{\infty} \mathcal{A}^i \mathcal{B}^T \mathcal{B} (\mathcal{A}^T)^i.$$

Finally, the output variance is given by $\sigma_w^2 = \mathcal{C}\Psi\mathcal{C}^T\sigma_v^2$, which completes the proof. □

2 Proof of Lemma 2

Proof. The integral is evaluated as

$$\frac{\sqrt{\pi}}{\sqrt{2}\kappa_2^2}(\boldsymbol{\theta_1}+\boldsymbol{\theta_3})\mathrm{erf}\left(\frac{\kappa_1+\tau}{\sqrt{2}\kappa_2}\right)-\frac{\boldsymbol{\theta_2}\kappa_2^2+\boldsymbol{\theta_3}(\kappa_1+\tau)}{{\kappa_2}^3}e^{-\frac{(\kappa_1+\tau)^2}{2\kappa_2^2}}+c,$$

where $\mathrm{erf}(\cdot)$ indicates the error function. As $\boldsymbol{\theta_1}=-\boldsymbol{\theta_3}$, the proof is complete. □

3 Proof of Proposition 1

Proof. After some math operation, the second derivative of the cost function is obtained as

$$\frac{\partial^2 J}{\partial \boldsymbol{\lambda}^{*\mathbf{2}}}=\left(\bar{H}\boldsymbol{\lambda}^*\mathbf{1}_{fg}^T+\mathbf{1}_{fg}^T\boldsymbol{\lambda}^*\bar{H}-2\bar{H}\boldsymbol{\lambda}^{*T}\mathbf{1}_{fg}\right)\frac{e^{-\frac{{w_{tp}^*}^2}{2\sigma_w^2}}}{2\sqrt{2\pi}\sigma_w^3}. \quad \text{(A.3)}$$

Here, the term $\frac{e^{-\frac{{w_{tp}^*}^2}{2\sigma_w^2}}}{2\sqrt{2\pi}\sigma_w^3}$ is non-negative. Substituting the optimal answer in $\bar{H}\boldsymbol{\lambda}^*\mathbf{1}_{fg}^T-\mathbf{1}_{fg}^T\boldsymbol{\lambda}^*\bar{H}-2\bar{H}\boldsymbol{\lambda}^{*T}\mathbf{1}_{fg}$ yields $\bar{H}(\mathbf{1}_{fg}\bar{H}^{-1}\mathbf{1}_{fg}^T)-\mathbf{1}_{fg}^T\mathbf{1}_{fg}$. To show that this is positive semi-definite we use Schur's complement lemma (see Ref. 2). According to this lemma we need to prove that the following statement holds

$$\begin{bmatrix}\bar{H}(\mathbf{1}_{fg}\bar{H}^{-1}\mathbf{1}_{fg}^T) & \mathbf{1}_{fg}^T\\ \mathbf{1}_{fg} & 1\end{bmatrix}\succeq 0. \quad \text{(A.4)}$$

Let us define $\Phi\triangleq\bar{H}(\mathbf{1}_{fg}\bar{H}^{-1}\mathbf{1}_{fg}^T)$. As $\bar{H}$ is positive definite, so is $\bar{H}^{-1}$, which yields $\mathbf{1}_{fg}\bar{H}^{-1}\mathbf{1}_{fg}^T$ 0. Hence, Φ is positive definite and invertible. The matrix in (A.4) can be decomposed as

$$\begin{bmatrix}\Phi & \mathbf{1}_{fg}^T\\ \mathbf{1}_{fg} & 1\end{bmatrix}=\begin{bmatrix}I\\ \mathbf{1}_{fg}\Phi^{-1}\end{bmatrix}\Phi\begin{bmatrix}I & \Phi^{-1}\mathbf{1}_{fg}^T\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 1-\mathbf{1}_{fg}\Phi^{-1}\mathbf{1}_{fg}^T\end{bmatrix}. \quad \text{(A.5)}$$

It can be verified that $1-\mathbf{1}_{fg}\Phi^{-1}\mathbf{1}_{fg}^T=0$. Considering the positive definiteness of Φ and based on the equation in (A.5) we conclude that (A.4) holds. □

4 Proof of Corollary 1

Proof. In case that we have no information from the plant and the controller, both u and y are modeled as random processes where the samples are independent and identically distributed. Moreover, in the conventional linear filters, u and y were not combined. So to interpret the conventional approach according to the proposed method we assume that $u(k)$ and $y(k)$ are the same. Now we have

$$y(k) = \nu(k) + b, \tag{A.6}$$

$$u(k) = \nu(k) + b, \tag{A.7}$$

where $\nu(k), k \in \{1, 2, \cdots\}$, follow a Gaussian distribution and b is defined in (4). By comparing (A.6) and (A.7) with (11) and (10) we determine $\boldsymbol{f} = 1$, $\boldsymbol{g} = 1$ and $\alpha(z^{-1})f(z^{-1}) - \beta(z^{-1})g(z^{-1}) = 1$. From Lemma 1 we find $\Psi = I$. Furthermore we obtain $H_f = G_g = I$ and hence $\bar{H} = \begin{bmatrix} I & I \\ I & I \end{bmatrix}$; thus the optimal filter is obtained as $\boldsymbol{\lambda}^* = c\ [0.5\ 0.5\ \cdots\ 0.5]$. This means that the optimal solution in the conventional framework is the case that all filter coefficients are equal. □

5 Proof of Lemma 3

Proof. The cost function J considering the optimal trip-point is given by

$$J = \eta_m \int_{-\infty}^{w_{\text{tp}}^*} \frac{1}{\sqrt{2\pi}\sigma_w} e^{-\frac{(\tau-\mu_w)^2}{2\sigma_w^2}} \mathrm{d}\tau + \eta_f \int_{w_{\text{tp}}^*}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_w} e^{-\frac{\tau^2}{2\sigma_w^2}} \mathrm{d}\tau. \tag{A.8}$$

As the alarm signal follows a Gaussian distribution, we have $\mu_{w_c} = c\mu_w$ and $\sigma_{w_c}^2 = c^2\sigma_{w_c}^2$, where μ_{w_c} and $\sigma_{w_c}^2$ are the mean and the variance of w_c, respectively. Furthermore, from (18) we have

$$w_{\text{tp}_c}^* = \frac{c\mu_w}{2} - \frac{c^2\sigma_w^2}{c\mu_w} \ln\left(\frac{\eta_f}{\eta_m}\right), \tag{A.9}$$

where $w^*_{\mathrm{tp}_c}$ corresponds to the optimal trip-point calculated for w_c. By comparing (18) and (A.9) we conclude that $w^*_{\mathrm{tp}_c} = cw^*_{\mathrm{tp}}$. Now J_c can be obtained as

$$J_c = \eta_m \int_{-\infty}^{cw^*_{\mathrm{tp}}} \frac{1}{\sqrt{2\pi}c\sigma_w} e^{-\frac{(\tau - c\mu_w)^2}{2c^2\sigma_w^2}} \mathrm{d}\tau + \eta_f \int_{cw^*_{\mathrm{tp}}}^{\infty} \frac{1}{\sqrt{2\pi}c\sigma_w} e^{-\frac{\tau^2}{2c^2\sigma_w^2}} \mathrm{d}\tau.$$

After some modification we have

$$J_c = \eta_m \int_{-\infty}^{cw^*_{\mathrm{tp}}} \frac{1}{\sqrt{2\pi}\sigma_w} e^{-\frac{(\frac{\tau}{c} - \mu_w)^2}{2\sigma_w^2}} \frac{1}{c} \mathrm{d}\tau + \eta_f \int_{cw^*_{\mathrm{tp}}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_w} e^{-\frac{(\frac{\tau}{c})^2}{2\sigma_w^2}} \frac{1}{c} \mathrm{d}\tau.$$

Substituting $\tau_c \triangleq \frac{\tau}{c}$ yields

$$J_c = \eta_m \int_{-\infty}^{w^*_{\mathrm{tp}}} \frac{1}{\sqrt{2\pi}\sigma_w} e^{-\frac{(\tau_c - \mu_w)^2}{2\sigma_w^2}} \mathrm{d}\tau_c + \eta_f \int_{w^*_{\mathrm{tp}}}^{\infty} \frac{1}{\sqrt{2\pi}\sigma_w} e^{-\frac{\tau_c^2}{2\sigma_w^2}} \mathrm{d}\tau_c. \tag{A.10}$$

By comparing (A.8) and (A.10) we can infer that $J_c = J$, and the proof is complete. $\square$

References

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