Supporting Information for: Modeling metasurfaces using discrete-space impulse response technique

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Supplementary Note 1: Relation between DSIR and angular transmission spectrum of non-diffractive periodic metasurfaces

Here we show that there is a Fourier transform relation between the DSIR of a non-diffractive periodic metasurface and its transmission angular spectrum. The Fourier transform of a discrete signal $t(n)$ is defined as[1]

$$\widetilde{t}(\omega) = \sum_{n=-\infty}^{\infty} t(n)\mathrm{e}^{j\omega n}, \tag{1}$$

and the inverse transform is given by

$$t(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \widetilde{t}(\omega)\mathrm{e}^{-j\omega n}\,\mathrm{d}\omega. \tag{2}$$

To find the DSIR, the metasurface is excited by an incident wave whose field on the input reference plane is given by $F_{\mathrm{in}}(x) = \mathrm{sinc}(x/\Lambda)$. The incident wave can be expressed as a sum of plane waves. To find the outgoing wave we find the response of the metasurface to these plane waves and superimpose them to form the outgoing wave. Using the Fourier transform of a sinc function, we can write

$$F_{\mathrm{in}}(x) = \mathrm{sinc}(\frac{x}{\Lambda}) = \frac{\Lambda}{2\pi}\int_{-\pi/\Lambda}^{\pi/\Lambda} \mathrm{e}^{-jk_x x}\,\mathrm{d}k_x, \tag{3}$$

which is a continuous sum over plane waves $\mathrm{e}^{-jk_x x}$ that are incident at angle θ where $\sin(\theta) = \frac{k_x}{n_1 k_0}$. Let $T(\theta)$ represents the complex-valued transmission coefficient of the metasurface for a plane wave incident at angle θ (i.e., the ratio of the tangential component of the transmitted field on output reference plane to that of the incident one on the input reference plane). The outgoing field can be written as a sum of plane waves whose amplitudes are modified by the transmission coefficient

$$F_{\text{out}}(x) = \frac{\Lambda}{2\pi}\int_{-\pi/\Lambda}^{\pi/\Lambda} T(\theta)\mathrm{e}^{-jk_x x}\,\mathrm{d}k_x. \tag{4}$$

The DSIR is obtained by sampling the outgoing wave

$$t(n) = F_{\text{out}}(n\Lambda) = \frac{\Lambda}{2\pi}\int_{-\pi/\Lambda}^{\pi/\Lambda} T(\theta)\mathrm{e}^{-jk_x n\Lambda}\,\mathrm{d}k_x = \frac{1}{2\pi}\int_{-\pi}^{\pi} T(\theta)\mathrm{e}^{-j\omega n}\,\mathrm{d}\omega \tag{5}$$

where we have defined $\omega = k_x\Lambda$. Comparing (5) and the inverse Fourier transform relation (2), we obtain

$$T(\theta) = \widetilde{t}(\omega) = \widetilde{t}(k_x\Lambda) = \widetilde{t}(n_1 k_0 \sin(\theta)\Lambda) = \widetilde{t}(2\pi\sin(\theta)\frac{\Lambda}{\lambda_1}) \tag{6}$$

Supplementary Note 2: Bound on the Euclidean norm of DSIR of non-diffractive periodic metasurfaces

Here we show that the Euclidean (L^2) norm of the DSIR of a non-diffractive periodic metasurface is bounded by one. According to the Parseval's theorem

$$\sum_{n=-\infty}^{\infty} |t(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widetilde{t}(\omega)|^2 \mathrm{d}\omega \tag{7}$$

According to (6) $\widetilde{t}(\omega) = T(\theta)$, and for passive metasurfaces the transmission coefficient modulus is smaller than unity, thus

$$\sum_{n=-\infty}^{\infty} |t(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |\widetilde{t}(\omega)|^2 \mathrm{d}\omega <= \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\omega = 1 \tag{8}$$

References

(1) Oppenheim, A. V. *Discrete-time signal processing*; Pearson Education India, 1999.