Supporting Information: Electron Transfer Methods in Open Systems

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Derivation of Eq. (11)

Here we derive exact equation-of-motion (EOM), Eq. (11), starting from definition of the probability in terms of Hubbard operator, Eq. (10). Writing Heisenberg EOM for the Hubbard operator in (10) and evaluating commutator for the model Hamiltonian (1)-(4) we get

$$\begin{aligned}\frac{d}{dt}P_S(t) &= i\langle[\hat{H};\hat{X}_{SS}](t)\rangle = i\sum_{K=L,R}\langle[\hat{V}_{MK};\hat{X}_{SS}](t)\rangle \\ &= 2\,\mathrm{Re}\sum_{S_3\in M}\sum_{K=L,R}\sum_{k\in K}\Big(V_{SS_3,k}\,G^<_{k,SS_3}(t,t) - V_{k,S_3S}\,G^<_{k,S_3S}(t,t)\Big)\end{aligned} \tag{S1}$$

where

$$G^<_{k,SS'}(t,t) \equiv i\langle\hat{X}^\dagger_{SS'}(t)\,\hat{c}_k(t)\rangle \tag{S2}$$

is equal time lesser projection of contour Green's function

$$G_{k,SS'}(\tau_1,\tau_2) \equiv -i\langle T_c\,\hat{c}_k(\tau_1)\,\hat{X}^\dagger_{SS'}(\tau_2)\rangle \tag{S3}$$

Integral form of the Dyson equation for (S3) is

$$G_{k,SS'}(\tau_1,\tau_2) = \sum_{S_1,S_2}\int_c d\tau' g_k(\tau_1,\tau')\,V_{k,S_1S_2}\,G_{S_1S_2,SS'}(\tau',\tau_2) \tag{S4}$$

where g_k is defined below Eq. (12) in the main text.

Lesser projection of (S4) taken at $t_1 = t_2 = t$ (t_1 and t_2 are physical times corresponding to contour variables τ_1 and τ_2, respectively) is

$$G^<_{k,SS'}(t,t) = \sum_{S_1,S_2}\int_{-\infty}^{+\infty} dt'\,\Big(g^r_k(t,t')\,V_{k,S_1S_2}\,G^<_{S_1S_2,SS'}(t',t) + g^<_k(t,t')\,V_{k,S_1S_2}\,G^a_{S_1S_2,SS'}(t',t)\Big) \tag{S5}$$

where r (a) superscript indicates retarded (advanced) projection. Using definition of the latter in terms of lesser and greater projections

$$g_k^r(t,t') \equiv \theta(t-t')\Big(g_k^>(t,t') - g_k^<(t,t')\Big) \tag{S6}$$

$$G^a_{S_1S_2,SS'}(t',t) \equiv \theta(t-t')\Big(G^<_{S_1S_2,SS'}(t',t) - G^>_{S_1S_2,SS'}(t',t)\Big) \tag{S7}$$

we get

$$G^<_{k,SS'}(t,t) = \sum_{S_1,S_2}\int_{-\infty}^{t} dt'\,\Big(g_k^>(t,t')\,V_{k,S_1S_2}\,G^<_{S_1S_2,SS'}(t',t) - g_k^<(t,t')\,V_{k,S_1S_2}\,G^>_{S_1S_2,SS'}(t',t)\Big) \tag{S8}$$

Finally, substituting (S8) into (S1) and using definition of self-energy given in Eq. (12) of the main text leads to exact EOM (11).

Expressions for Rates

Here we give explicit expressions for second and fourth order transfer rates: first in terms of locators, spectral weights and correlation functions; after that, result of substitution of zero order expressions for the latter. results of Refs. 1,2 for $W^{(n)}_{S_f\leftarrow S_i}$ rate is obtained by setting $P_{S_i}=1$ and all other probabilities to 0.

Second Order Rates

The only diagram contributing to second order rate is shown in Fig. 1a of the main text.

First and fourth terms in the right side of Eq. (11) in the main text

$$2\,\mathrm{Re}\sum_{K\in\{L,R\}}\int_{-\infty}^{t}dt'$$
$$\Big(\sigma^{K\,>}_{S_fS_i,S_fS_i}(t-t')\,g^{<}_{S_fS_i}(t'-t)\,\langle\hat{F}_{S_fS_i,S_fS_i}(t)\rangle$$
$$+\,\sigma^{K\,<}_{S_iS_f,S_iS_f}(t-t')\,g^{>}_{S_iS_f}(t'-t)\,\langle\hat{F}_{S_iS_f,S_iS_f}(t)\rangle\Big)$$

contribute to $W^{(2)}_{S_f\leftarrow S_i}\,P_{S_i}$. Explicit expression for the rate is

$$W^{(2)}_{S_f\leftarrow S_i}=i\sum_{K\in\{L,R\}}\Big(\sigma^{K\,>}_{S_fS_i,S_fS_i}(E_{S_i}-E_{S_f})-\sigma^{K\,<}_{S_iS_f,S_iS_f}(E_{S_f}-E_{S_i})\Big)$$

Second and third terms in the right side of Eq. (11) in the main text

$$-\,2\,\mathrm{Re}\sum_{S}\sum_{K\in\{L,R\}}\int_{-\infty}^{t}dt'$$
$$\Big(\sigma^{K\,>}_{SS_i,SS_i}(t-t')\,g^{<}_{SS_i}(t'-t)\,\langle\hat{F}_{SS_i,SS_i}(t)\rangle$$
$$+\,\sigma^{K\,<}_{S_iS,S_iS}(t-t')\,g^{>}_{S_iS}(t'-t)\,\langle\hat{F}_{S_iS,S_iS}(t)\rangle\Big)$$

contribute to $W^{(2)}_{S_i\leftarrow S_i}\,P_{S_i}$. Explicit expression for the rate is

$$W^{(2)}_{S_i\leftarrow S_i}=-i\sum_{S}\sum_{K\in\{L,R\}}\Big(\sigma^{K\,>}_{SS_i,SS_i}(E_{S_i}-E_S)-\sigma^{K\,<}_{S_iS,S_iS}(E_S-E_{S_i})\Big)$$

It is easy to check that

$$\sum_{S_f}W^{(2)}_{S_f\leftarrow S_i}=0$$

Fourth Order Rates

Contributions are classified by projection in Fig. 3 and diagram in Fig. 1 of the main text. In the expressions below

$$B_{SS'}(\tau,\tau') \equiv -i\langle \hat{X}_{SS'}(\tau)\,\hat{X}^{\dagger}_{SS'}(\tau')\rangle \qquad N_S = N_{S'}$$

$$D_{SS'}(\tau,\tau') \equiv -i\langle \hat{X}_{SS'}(\tau)\,\hat{X}^{\dagger}_{SS'}(\tau')\rangle \qquad N_S + 2 = N_{S'}$$

A.(0).(s) projections

This contribution is of the type $W^{(4)}_{S_i\leftarrow S_i}\,P_{S_i}$

Diagram (c)

$$\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2,S_3}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt'\int_{-\infty}^{t'}dt_2\\
&\Big(\,\sigma^{K_1<}_{S_iS_1,S_3S_2}(t-t')\,g^{>}_{S_3S_2}(t'-t_1)\,g^{<}_{S_3S_i}(t_1-t_2)\,\sigma^{K_2>}_{S_3S_i,S_2S_1}(t_2-t_1)\,g^{>}_{S_iS_1}(t_1-t)\langle\hat{F}_{S_iS_1,S_iS_1}(t)\rangle\\
&-\sigma^{K_1>}_{S_1S_i,S_2S_3}(t-t')\,g^{<}_{S_2S_3}(t'-t_1)\,g^{>}_{S_iS_3}(t_1-t_2)\,\sigma^{K_2<}_{S_iS_3,S_1S_2}(t_2-t_1)\,g^{<}_{S_1S_i}(t_1-t)\langle\hat{F}_{S_1S_i,S_1S_i}(t)\rangle\\
&+\sigma^{K_1>}_{S_1S_i,S_2S_3}(t-t')\,g^{>}_{S_2S_3}(t'-t_2)\,g^{<}_{S_2S_1}(t_2-t_1)\,\sigma^{K_2>}_{S_2S_1,S_3S_i}(t_1-t_2)\,g^{<}_{S_1S_i}(t_2-t)\langle\hat{F}_{S_1S_i,S_1S_i}(t)\rangle\\
&-\sigma^{K_1<}_{S_iS_1,S_3S_2}(t-t')\,g^{<}_{S_3S_2}(t'-t_2)\,g^{>}_{S_1S_2}(t_2-t_1)\,\sigma^{K_2<}_{S_1S_2,S_iS_3}(t_1-t_2)\,g^{>}_{S_iS_1}(t_2-t)\langle\hat{F}_{S_iS_1,S_iS_1}(t)\rangle\Big)
\end{aligned}$$

Diagram (d)

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2,S_3}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt'\int_{-\infty}^{t'}dt_2\\
&\Big(\,\sigma^{K_1\,>}_{S_1S_i,S_2S_3}(t-t')\,g^{>}_{S_2S_3}(t'-t_2)\,\sigma^{K_2\,<}_{S_iS_3,S_1S_2}(t_2-t_1)\,g^{>}_{S_1S_2}(t_1-t_2)\,g^{<}_{S_1S_i}(t_2-t)\langle\hat F_{S_1S_i,S_1S_i}(t)\rangle\\
&-\sigma^{K_1\,<}_{S_iS_1,S_3S_2}(t-t')\,g^{<}_{S_3S_2}(t'-t_2)\,\sigma^{K_2\,>}_{S_3S_i,S_2S_1}(t_2-t_1)\,g^{<}_{S_2S_1}(t_1-t_2)\,g^{>}_{S_iS_1}(t_2-t)\,\langle\hat F_{S_iS_1,S_iS_1}(t)\rangle\\
&+\sigma^{K_1\,<}_{S_iS_1,S_3S_2}(t-t')\,g^{>}_{S_3S_2}(t'-t_1)\,\sigma^{K_2\,<}_{S_1S_2,S_iS_3}(t_1-t_2)\,g^{>}_{S_iS_3}(t_2-t_1)\,g^{>}_{S_iS_1}(t_1-t)\langle\hat F_{S_iS_1,S_iS_1}(t)\rangle\\
&-\sigma^{K_1\,>}_{S_1S_i,S_2S_3}(t-t')\,g^{<}_{S_2S_3}(t'-t_1)\,\sigma^{K_2\,>}_{S_2S_1,S_3S_i}(t_1-t_2)\,g^{<}_{S_3S_i}(t_2-t_1)\,g^{<}_{S_1S_i}(t_1-t)\langle\hat F_{S_1S_i,S_1S_i}(t)\rangle\Big)
\end{aligned}
$$

Diagram (g)

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2,S_3}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt'\int_{-\infty}^{t'}dt_2\\
&\Big(\sigma^{K_1\,<}_{S_iS_1,S_3S_2}(t-t')\,g^{<}_{S_3S_2}(t'-t_2)\,\sigma^{K_2\,>}_{S_3S_i,S_2S_1}(t_2-t_1)\,g^{>}_{S_2S_1}(t_1-t)\,B^{<}_{S_2S_i}(t-t_2)\\
&+\sigma^{K_1\,>}_{S_1S_i,S_2S_3}(t-t')\,g^{>}_{S_2S_3}(t'-t_2)\,\sigma^{K_2\,<}_{S_iS_3,S_1S_2}(t_2-t_1)\,g^{<}_{S_1S_2}(t_1-t)\,B^{>}_{S_iS_2}(t-t_2)\Big)
\end{aligned}
$$

Diagram (h)

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2,S_3}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt'\int_{-\infty}^{t'}dt_2\\
&\Big(\sigma^{K_1\,<}_{S_iS_1,S_3S_2}(t-t')\,g^{<}_{S_3S_2}(t'-t_2)\,D^{>}_{S_iS_2}(t_2-t)\,g^{<}_{S_1S_2}(t-t_1)\,\sigma^{K_2\,<}_{S_1S_2,S_iS_3}(t_1-t_2)\\
&+\sigma^{K_1\,>}_{S_1S_i,S_2S_3}(t-t')\,g^{>}_{S_2S_3}(t'-t_2)\,D^{<}_{S_2S_i}(t_2-t)\,g^{>}_{S_2S_1}(t-t_1)\,\sigma^{K_2\,>}_{S_2S_1,S_3S_i}(t_1-t_2)\Big)
\end{aligned}
$$

Resulting zero-order expression

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2,S_3}\sum_{K_1,K_2\in\{L,R\}}\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\left(\frac{P_{S_i}\,P_{S_3}}{P_{S_i}+P_{S_3}}+P_{S_2}\right)\frac{P_{S_i}}{P_{S_2}+P_{S_3}}\\
&\Bigg[\frac{\Gamma^{K_1}_{S_iS_1,S_3S_2}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_3S_i,S_2S_1}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_3}-E_{S_i}+\omega_2+i0)(E_{S_2}-E_{S_i}-\omega_1+\omega_2+i0)(E_{S_1}-E_{S_i}-\omega_1+i0)}\\
&+\frac{\Gamma^{K_1}_{S_iS_1,S_3S_2}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_1S_2,S_iS_3}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_3}-E_{S_i}-\omega_2+i0)(E_{S_2}-E_{S_i}-\omega_1-\omega_2+i0)(E_{S_1}-E_{S_i}-\omega_1+i0)}\\
&+\frac{\Gamma^{K_1}_{S_1S_i,S_2S_3}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_2S_1,S_3S_i}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_i}-E_{S_3}-\omega_2+i0)(E_{S_i}-E_{S_2}-\omega_1-\omega_2+i0)(E_{S_i}-E_{S_1}-\omega_1+i0)}\\
&+\frac{\Gamma^{K_1}_{S_1S_i,S_2S_3}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_iS_3,S_1S_2}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_i}-E_{S_3}+\omega_2+i0)(E_{S_i}-E_{S_2}-\omega_1+\omega_2+i0)(E_{S_i}-E_{S_1}-\omega_1+i0)}\Bigg]
\end{aligned}
$$

A.(1).(s) projections

This contribution is of the type $W^{(4)}_{S_f\leftarrow S_i}\,P_{S_i}$

Diagram (c)

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt'\int_{-\infty}^{t'}dt_2\\
&\Big(\,\sigma^{K_1\,>}_{S_fS_i,S_2S_1}(t-t')\,g^{>}_{S_2S_1}(t'-t_2)\,g^{<}_{S_2S_f}(t_2-t_1)\,\sigma^{K_2\,>}_{S_2S_f,S_1S_i}(t_1-t_2)\,g^{<}_{S_fS_i}(t_2-t)\langle\hat{F}_{S_fS_i,S_fS_i}(t)\rangle\\
&-\sigma^{K_1\,<}_{S_iS_f,S_1S_2}(t-t')\,g^{<}_{S_1S_2}(t'-t_2)\,g^{>}_{S_fS_2}(t_2-t_1)\,\sigma^{K_2\,<}_{S_fS_2,S_iS_1}(t_1-t_2)\,g^{>}_{S_iS_f}(t_2-t)\langle\hat{F}_{S_iS_f,S_iS_f}(t)\rangle\\
&+\sigma^{K_1\,<}_{S_iS_f,S_1S_2}(t-t')\,g^{>}_{S_1S_2}(t'-t_1)\,g^{<}_{S_1S_i}(t_1-t_2)\,\sigma^{K_2\,>}_{S_1S_i,S_2S_f}(t_2-t_1)\,g^{>}_{S_iS_f}(t_1-t)\langle\hat{F}_{S_iS_f,S_iS_f}(t)\rangle\\
&-\sigma^{K_1\,>}_{S_fS_i,S_2S_1}(t-t')\,g^{<}_{S_2S_1}(t'-t_1)\,g^{>}_{S_iS_1}(t_1-t_2)\,\sigma^{K_2\,<}_{S_iS_1,S_fS_2}(t_2-t_1)\,g^{<}_{S_fS_i}(t_1-t)\langle\hat{F}_{S_fS_i,S_fS_i}(t)\rangle\Big)
\end{aligned}
$$

Diagram (d)

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt'\int_{-\infty}^{t'}dt_2\\
&\Big(\;\sigma^{K_1<}_{S_iS_f,S_1S_2}(t-t')\,g^{>}_{S_1S_2}(t'-t_1)\,\sigma^{K_2<}_{S_fS_2,S_iS_1}(t_1-t_2)\,g^{>}_{S_iS_1}(t_2-t_1)\,g^{>}_{S_iS_f}(t_1-t)\langle\hat{F}_{S_iS_f,S_iS_f}(t)\rangle\\
&-\sigma^{K_1>}_{S_fS_i,S_2S_1}(t-t')\,g^{<}_{S_2S_1}(t'-t_1)\,\sigma^{K_2>}_{S_2S_f,S_1S_i}(t_1-t_2)\,g^{<}_{S_1S_i}(t_2-t_1)\,g^{<}_{S_fS_i}(t_1-t)\langle\hat{F}_{S_fS_i,S_f,S_i}(t)\rangle\\
&+\sigma^{K_1>}_{S_fS_i,S_2S_1}(t-t')\,g^{>}_{S_2S_1}(t'-t_2)\,\sigma^{K_2<}_{S_iS_1,S_fS_2}(t_2-t_1)\,g^{>}_{S_fS_2}(t_1-t_2)\,g^{<}_{S_fS_i}(t_2-t)\langle\hat{F}_{S_fS_i,S_fS_i}(t)\rangle\\
&-\sigma^{K_1<}_{S_iS_f,S_1S_2}(t-t')\,g^{<}_{S_1S_2}(t'-t_2)\,\sigma^{K_2>}_{S_1S_i,S_2S_f}(t_2-t_1)\,g^{<}_{S_2S_f}(t_1-t_2)\,g^{>}_{S_iS_f}(t_2-t)\langle\hat{F}_{S_iS_f,S_iS_f}(t)\rangle\Big)
\end{aligned}
$$

Diagram (g)

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt'\int_{-\infty}^{t'}dt_2\\
&\Big(\;\sigma^{K_1>}_{S_fS_i,S_2S_1}(t-t')\,g^{>}_{S_2S_1}(t'-t_2)\,\sigma^{K_2<}_{S_iS_1,S_fS_2}(t_2-t_1)\,g^{<}_{S_fS_2}(t_1-t)\,B^{>}_{S_iS_2}(t-t_2)\\
&+\sigma^{K_1<}_{S_iS_f,S_1S_2}(t-t')\,g^{<}_{S_1S_2}(t'-t_2)\,\sigma^{K_2>}_{S_1S_i,S_2S_f}(t_2-t_1)\,g^{>}_{S_2S_f}(t_1-t)\,B^{<}_{S_2S_i}(t-t_2)\Big)
\end{aligned}
$$

Diagram (h)

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt'\int_{-\infty}^{t'}dt_2\\
&\Big(\sigma^{K_1>}_{S_fS_i,S_2S_1}(t-t')\,g^{>}_{S_2S_1}(t'-t_2)\,D^{<}_{S_2S_i}(t_2-t)\,g^{>}_{S_2S_f}(t-t_1)\,\sigma^{K_2>}_{S_2S_f,S_1S_i}(t_1-t_2)\\
&+\sigma^{K_1<}_{S_iS_f,S_1S_2}(t-t')\,g^{<}_{S_1S_2}(t'-t_2)\,D^{>}_{S_iS_2}(t_2-t)\,g^{<}_{S_fS_2}(t-t_1)\,\sigma^{K_2<}_{S_fS_2,S_iS_1}(t_1-t_2)\Big)
\end{aligned}
$$

Resulting zero-order expression

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\left(P_{S_2}+\frac{P_{S_i}P_{S_1}}{P_{S_i}+P_{S_1}}\right)\frac{P_{S_i}}{P_{S_1}+P_{S_2}}\\
&\Bigg[\frac{\Gamma^{K_1}_{S_fS_i,S_2S_1}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_2S_f,S_1S_i}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_i}-E_{S_1}-\omega_2+i0)(E_{S_i}-E_{S_2}-\omega_1-\omega_2+i0)(E_{S_i}-E_{S_f}-\omega_1+i0)}\\
&+\frac{\Gamma^{K_1}_{S_fS_i,S_2S_1}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_iS_1,S_fS_2}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_i}-E_{S_1}+\omega_2+i0)(E_{S_i}-E_{S_2}-\omega_1+\omega_2+i0)(E_{S_i}-E_{S_f}-\omega_1+i0)}\\
&+\frac{\Gamma^{K_1}_{S_iS_f,S_1S_2}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_1S_i,S_2S_f}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_1}-E_{S_i}+\omega_2+i0)(E_{S_2}-E_{S_i}-\omega_1+\omega_2+i0)(E_{S_f}-E_{S_i}-\omega_1+i0)}\\
&+\frac{\Gamma^{K_1}_{S_iS_f,S_1S_2}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_fS_2,S_iS_1}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_1}-E_{S_i}-\omega_2+i0)(E_{S_2}-E_{S_i}-\omega_1-\omega_2+i0)(E_{S_f}-E_{S_i}-\omega_1+i0)}\Bigg]
\end{aligned}
$$

A.(1).(t) projections

This contribution is of the type $W^{(4)}_{S_f\leftarrow S_i}\,P_{S_i}$

Diagram (g)

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt'\int_{-\infty}^{t}dt_2\\
&\Big(\sigma^{K_1\,<}_{S_fS_2,S_iS_1}(t-t')\,g^{>}_{S_iS_1}(t'-t_1)\,\sigma^{K_2\,>}_{S_2S_1,S_fS_i}(t_1-t_2)\,g^{<}_{S_fS_i}(t_2-t)\,B^{<}_{S_2S_i}(t-t_1)\\
&+\sigma^{K_1\,>}_{S_2S_f,S_iS_1}(t-t')\,g^{<}_{S_1S_i}(t'-t_1)\,\sigma^{K_2\,<}_{S_1S_2,S_iS_f}(t_1-t_2)\,g^{>}_{S_iS_f}(t_2-t)\,B^{>}_{S_iS_2}(t-t_1)\Big)
\end{aligned}
$$

Diagram (h)

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt'\int_{-\infty}^{t}dt_2\\
&\Big(\sigma^{K_1\,<}_{S_fS_2,S_iS_1}(t-t')\,g^{>}_{S_iS_1}(t'-t_1)\,D^{>}_{S_iS_2}(t_1-t)\,g^{>}_{S_iS_f}(t-t_2)\,\sigma^{K_2\,<}_{S_iS_f,S_1S_2}(t_2-t_1)\\
&+\sigma^{K_1\,>}_{S_2S_f,S_1S_i}(t-t')\,g^{<}_{S_1S_i}(t'-t_1)\,D^{<}_{S_2S_i}(t_1-t)\,g^{<}_{S_fS_i}(t-t_2)\,\sigma^{K_2\,>}_{S_fS_i,S_2S_1}(t_2-t_1)\Big)
\end{aligned}
$$

Resulting zero-order expression

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\,\frac{P_{S_i}^3}{(P_{S_i}+P_{S_f})(P_{S_i}+P_{S_1})}\\
&\Bigg[\frac{\Gamma^{K_1}_{S_fS_2,S_iS_1}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_2S_1,S_fS_i}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_1}-E_{S_i}-\omega_1+i0)(E_{S_2}-E_{S_i}-\omega_1+\omega_2+i0)(E_{S_i}-E_{S_f}-\omega_2+i0)}\\
&+\frac{\Gamma^{K_1}_{S_fS_2,S_iS_1}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_iS_f,S_1S_2}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_1}-E_{S_i}-\omega_1+i0)(E_{S_2}-E_{S_i}-\omega_1-\omega_2+i0)(E_{S_i}-E_{S_f}+\omega_2+i0)}\\
&+\frac{\Gamma^{K_1}_{S_2S_f,S_1S_i}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_fS_i,S_2S_1}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_i}-E_{S_1}-\omega_1+i0)(E_{S_i}-E_{S_2}-\omega_1-\omega_2+i0)(E_{S_f}-E_{S_i}+\omega_2+i0)}\\
&+\frac{\Gamma^{K_1}_{S_2S_f,S_iS_1}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_1S_2,S_iS_f}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_i}-E_{S_1}-\omega_1+i0)(E_{S_i}-E_{S_2}-\omega_1+\omega_2+i0)(E_{S_f}-E_{S_i}-\omega_2+i0)}\Bigg]
\end{aligned}
$$

A.(2).(t) projections

This contribution is of the type $W^{(4)}_{S_f\leftarrow S_i}\,P_{S_i}$

Diagram (g)

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt'\int_{-\infty}^{t}dt_2\\
&\Big(\,\sigma^{K_1\,>}_{S_fS_2,S_1S_i}(t-t')\,g^{<}_{S_1S_i}(t'-t_1)\,\sigma^{K_2\,<}_{S_1S_f,S_iS_2}(t_1-t_2)\,g^{>}_{S_iS_2}(t_2-t)\,B^{>}_{S_iS_f}(t-t_1)\\
&+\sigma^{K_1\,<}_{S_2S_f,S_iS_1}(t-t')\,g^{>}_{S_iS_1}(t'-t_1)\,\sigma^{K_2\,>}_{S_fS_1,S_2S_i}(t_1-t_2)\,g^{<}_{S_2S_i}(t_2-t)\,B^{<}_{S_fS_i}(t-t_1)\Big)
\end{aligned}
$$

Diagram (h)

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt'\int_{-\infty}^{t}dt_2\\
&\Big(\,\sigma^{K_1\,>}_{S_fS_2,S_1S_i}(t-t')\,g^{<}_{S_1S_i}(t'-t_1)\,D^{<}_{S_fS_i}(t_1-t)\,g^{<}_{S_2S_i}(t-t_2)\,\sigma^{K_2\,>}_{S_2S_i,S_fS_1}(t_2-t_1)\\
&+\sigma^{K_1\,<}_{S_2S_f,S_iS_1}(t-t')\,g^{>}_{S_iS_1}(t'-t_1)\,D^{>}_{S_iS_f}(t_1-t)\,g^{>}_{S_iS_2}(t-t_2)\,\sigma^{K_2\,<}_{S_iS_2,S_1S_f}(t_2-t_1)\Big)
\end{aligned}
$$

Resulting zero-order expression

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\,\frac{P_{S_i}^3}{(P_{S_i}+P_{S_1})(P_{S_i}+P_{S_2})}\\
&\Bigg[\frac{\Gamma^{K_1}_{S_fS_2,S_1S_i}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_2S_i,S_fS_1}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_i}-E_{S_1}-\omega_1+i0)(E_{S_i}-E_{S_f}-\omega_1-\omega_2+i0)(E_{S_2}-E_{S_i}+\omega_2+i0)}\\
&+\frac{\Gamma^{K_1}_{S_fS_2,S_1S_i}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_1S_f,S_iS_2}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_i}-E_{S_1}-\omega_1+i0)(E_{S_i}-E_{S_f}-\omega_1+\omega_2+i0)(E_{S_2}-E_{S_i}-\omega_2+i0)}\\
&+\frac{\Gamma^{K_1}_{S_2S_f,S_iS_1}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_iS_2,S_1S_f}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_1}-E_{S_i}-\omega_1+i0)(E_{S_f}-E_{S_i}-\omega_1-\omega_2+i0)(E_{S_i}-E_{S_2}+\omega_2+i0)}\\
&+\frac{\Gamma^{K_1}_{S_2S_f,S_iS_1}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_fS_1,S_2S_i}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_1}-E_{S_i}-\omega_1+i0)(E_{S_f}-E_{S_i}-\omega_1+\omega_2+i0)(E_{S_i}-E_{S_2}-\omega_2+i0)}\Bigg]
\end{aligned}
$$

B.(0).(s) projections

This contribution is of the type $W^{(4)}_{S_i\leftarrow S_i}\,P_{S_i}$

Diagram (b)

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2,S_3}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt_2\int_{-\infty}^{t_2}dt_1\\
&\Big(\sigma^{K_1\,>}_{S_1S_i,S_1S_2}(t-t')\,g^{<}_{S_1S_2}(t'-t)\,g^{>}_{S_iS_3}(t-t_1)\,\langle\hat{F}_{S_iS_3,S_iS_3}(t_1)\rangle\sigma^{K_2\,<}_{S_iS_3,S_2S_3}(t_1-t_2)\,g^{<}_{S_2S_3}(t_2-t)\\
&-\sigma^{K_1\,<}_{S_iS_1,S_2S_1}(t-t')\,g^{>}_{S_2S_1}(t'-t)\,g^{<}_{S_3S_i}(t-t_1)\,\langle\hat{F}_{S_3S_i,S_3S_i}(t_1)\rangle\,\sigma^{K_2\,>}_{S_3S_i,S_3S_2}(t_1-t_2)\,g^{>}_{S_3S_2}(t_2-t)\Big)
\end{aligned}
$$

Diagram (c)

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2,S_3}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt_2\int_{-\infty}^{t_2}dt_1\\
&\Big(\sigma^{K_1\,<}_{S_iS_1,S_2S_1}(t-t')\,g^{<}_{S_2S_1}(t'-t_2)\,g^{<}_{S_3S_i}(t_2-t_1)\,\sigma^{K_2\,>}_{S_3S_i,S_3S_2}(t_1-t_2)\,g^{>}_{S_iS_1}(t_2-t)\,\langle\hat{F}_{S_iS_1,S_iS_1}(t)\rangle\\
&-\sigma^{K_1\,>}_{S_1S_i,S_1S_2}(t-t')\,g^{>}_{S_1S_2}(t'-t_2)\,g^{>}_{S_iS_3}(t_2-t_1)\,\sigma^{K_2\,<}_{S_iS_3,S_2S_3}(t_1-t_2)\,g^{<}_{S_1S_i}(t_2-t)\,\langle\hat{F}_{S_1S_i,S_1S_i}(t)\rangle\Big)
\end{aligned}
$$

Diagram (d)

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2,S_3}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt_2\int_{-\infty}^{t_2}dt_1\\
&\Big(\,\sigma^{K_1<}_{S_iS_1,S_2S_1}(t-t')\,g^{<}_{S_2S_1}(t'-t_2)\,\sigma^{K_2<}_{S_2S_3,S_iS_3}(t_2-t_1)\,g^{>}_{S_iS_3}(t_1-t_2)\,g^{>}_{S_iS_1}(t_2-t)\,\langle\hat{F}_{S_iS_1,S_iS_1}(t)\rangle\\
&-\sigma^{K_1>}_{S_1S_i,S_1S_2}(t-t')\,g^{>}_{S_1S_2}(t'-t_2)\,\sigma^{K_2>}_{S_3S_2,S_3S_i}(t_2-t_1)\,g^{<}_{S_3S_i}(t_1-t_2)\,g^{<}_{S_1S_i}(t_2-t)\,\langle\hat{F}_{S_1S_i,S_1S_i}(t)\rangle\Big)
\end{aligned}
$$

Diagram (i)

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2,S_3}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt_2\int_{-\infty}^{t_2}dt_1\\
&\Big(\,\sigma^{K_1>}_{S_1S_i,S_1S_2}(t-t')\,g^{<}_{S_1S_2}(t'-t)\,B^{>}_{S_iS_2}(t-t_2)\,\sigma^{K_2>}_{S_3S_2,S_3S_i}(t_2-t_1)\,g^{<}_{S_3S_i}(t_1-t_2)\\
&+\sigma^{K_1<}_{S_iS_1,S_2S_1}(t-t')\,g^{>}_{S_2S_1}(t'-t)\,B^{<}_{S_2S_i}(t-t_2)\,\sigma^{K_2<}_{S_2S_3,S_iS_3}(t_2-t_1)\,g^{>}_{S_iS_3}(t_1-t_2)\\
&+\sigma^{K_1<}_{S_iS_1,S_2S_1}(t-t')\,g^{>}_{S_2S_1}(t'-t)\,B^{<}_{S_2S_i}(t-t_1)\,\sigma^{K_2>}_{S_3S_i,S_3S_2}(t_1-t_2)\,g^{<}_{S_3S_2}(t_2-t_1)\\
&+\sigma^{K_1>}_{S_1S_i,S_1S_2}(t-t')\,g^{<}_{S_1S_2}(t'-t)\,B^{>}_{S_iS_2}(t-t_1)\,\sigma^{K_2<}_{S_iS_3,S_2S_3}(t_1-t_2)\,g^{>}_{S_2S_3}(t_2-t_1)\Big)
\end{aligned}
$$

Resulting zero-order expression

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2,S_3}\sum_{K_1,K_2\in\{L,R\}}\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\\
&\Bigg[\quad\left(\frac{\Gamma^{K_1}_{S_iS_1,S_2S_1}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_3S_i,S_3S_2}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_3}-E_{S_i}+\omega_2+i0)(E_{S_2}-E_{S_i}+i0)(E_{S_1}-E_{S_i}-\omega_1+i0)}\right.\\
&\quad\left.+\frac{\Gamma^{K_1}_{S_1S_i,S_1S_2}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_iS_3,S_2S_3}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_i}-E_{S_3}+\omega_2+i0)(E_{S_i}-E_{S_2}+i0)(E_{S_i}-E_{S_1}-\omega_1+i0)}\right)\\
&\quad\times\left(P_{S_2}+\frac{P_{S_i}P_{S_1}}{P_{S_i}+P_{S_3}}\right)\frac{P_{S_i}}{P_{S_1}+P_{S_2}}\\
&+\quad\left(\frac{\Gamma^{K_1}_{S_iS_1,S_2S_1}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_2S_3,S_iS_3}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_3}-E_{S_i}-\omega_2+i0)(E_{S_2}-E_{S_i}+i0)(E_{S_1}-E_{S_i}-\omega_1+i0)}\right.\\
&\quad\left.+\frac{\Gamma^{K_1}_{S_1S_i,S_1S_2}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_3S_2,S_3S_i}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_i}-E_{S_3}-\omega_2+i0)(E_{S_i}-E_{S_2}+i0)(E_{S_i}-E_{S_1}-\omega_1+i0)}\right)\\
&\quad\times\frac{P^2_{S_i}}{P_{S_i}+P_{S_3}}\Bigg]
\end{aligned}
$$

B.(1).(s) projections

This contribution is of the type $W^{(4)}_{S_f\leftarrow S_i}\,P_{S_i}$

Diagram (b)

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt_2\int_{-\infty}^{t_2}dt_1\\
&\Big(\;\sigma^{K_1\,<}_{S_iS_f,S_2S_f}(t-t')\,g^{>}_{S_2S_f}(t'-t)\,g^{<}_{S_1S_i}(t-t_1)\,\langle\hat F_{S_1S_i,S_1S_i}(t_1)\rangle\,\sigma^{K_2\,>}_{S_1S_i,S_1S_2}(t_1-t_2)\,g^{>}_{S_1S_2}(t_2-t)\\
&-\sigma^{K_1\,>}_{S_fS_i,S_fS_2}(t-t')\,g^{<}_{S_fS_2}(t'-t)\,g^{>}_{S_iS_1}(t-t_1)\,\langle\hat F_{S_iS_1,S_iS_1}(t_1)\rangle\,\sigma^{K_2\,<}_{S_iS_1,S_2S_1}(t_1-t_2)\,g^{<}_{S_2S_1}(t_2-t)\Big)
\end{aligned}
$$

Diagram (c)

$$2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt_2\int_{-\infty}^{t_2}dt_1$$
$$\Big(\,\sigma^{K_1\,>}_{S_fS_i,S_fS_2}(t-t')\,g^{>}_{S_fS_2}(t'-t_2)\,g^{>}_{S_iS_1}(t_2-t_1)\,\sigma^{K_2\,<}_{S_iS_1,S_2S_1}(t_1-t_2)\,g^{<}_{S_fS_i}(t_2-t)\langle\hat{F}_{S_fS_i,S_fS_i}(t)\rangle$$
$$-\,\sigma^{K_1\,<}_{S_iS_f,S_2S_f}(t-t')\,g^{<}_{S_2S_f}(t'-t_2)\,g^{<}_{S_1S_i}(t_2-t_1)\,\sigma^{K_2\,>}_{S_1S_i,S_1S_2}(t_1-t_2)\,g^{>}_{S_iS_f}(t_2-t)\langle\hat{F}_{S_iS_f,S_iS_f}(t)\rangle\Big)$$

Diagram (d)

$$2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt_2\int_{-\infty}^{t_2}dt_1$$
$$\Big(\,\sigma^{K_1\,>}_{S_fS_i,S_fS_2}(t-t')\,g^{>}_{S_fS_2}(t'-t_2)\,\sigma^{K_2\,>}_{S_1S_2,S_1S_i}(t_2-t_1)\,g^{<}_{S_1S_i}(t_1-t_2)\,g^{<}_{S_fS_i}(t_2-t)\,\langle\hat{F}_{S_fS_i,S_fS_i}(t)\rangle$$
$$-\,\sigma^{K_1\,<}_{S_iS_f,S_2S_f}(t-t')\,g^{<}_{S_2S_f}(t'-t_2)\,\sigma^{K_2\,<}_{S_2S_1,S_iS_1}(t_2-t_1)\,g^{>}_{S_iS_1}(t_1-t_2)\,g^{>}_{S_iS_f}(t_2-t)\langle\hat{F}_{S_iS_f,S_iS_f}(t)\rangle\Big)$$

Diagram (i)

$$2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt_2\int_{-\infty}^{t_2}dt_1$$
$$\Big(\,\sigma^{K_1\,>}_{S_fS_i,S_fS_2}(t-t')\,g^{<}_{S_fS_2}(t'-t)\,B^{>}_{S_iS_2}(t-t_1)\,\sigma^{K_2\,<}_{S_iS_1,S_2S_1}(t_1-t_2)\,g^{>}_{S_2S_1}(t_2-t_1)$$
$$+\,\sigma^{K_1\,<}_{S_iS_f,S_2S_f}(t-t')\,g^{>}_{S_2S_f}(t'-t)\,B^{<}_{S_2S_i}(t-t_1)\,\sigma^{K_2\,>}_{S_1S_i,S_1S_2}(t_1-t_2)\,g^{<}_{S_1S_2}(t_2-t_1)$$
$$+\,\sigma^{K_1\,>}_{S_fS_i,S_fS_2}(t-t')\,g^{<}_{S_fS_2}(t'-t)\,B^{>}_{S_iS_2}(t-t_2)\,\sigma^{K_2\,>}_{S_1S_2,S_1S_i}(t_2-t_1)\,g^{<}_{S_1S_i}(t_1-t_2)$$
$$+\,\sigma^{K_1\,<}_{S_iS_f,S_2S_f}(t-t')\,g^{>}_{S_2S_f}(t'-t)\,B^{<}_{S_2S_i}(t-t_2)\,\sigma^{K_2\,<}_{S_2S_1,S_iS_1}(t_2-t_1)\,g^{>}_{S_iS_1}(t_1-t_2)\Big)$$

Resulting zero-order expression

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\\
&\Bigg[\quad\Bigg(\frac{\Gamma^{K_1}_{S_fS_i,S_fS_2}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_1S_2,S_1S_i}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_i}-E_{S_1}-\omega_2+i0)(E_{S_i}-E_{S_2}+i0)(E_{S_i}-E_{S_f}-\omega_1+i0)}\\
&\quad+\frac{\Gamma^{K_1}_{S_iS_f,S_2S_f}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_2S_1,S_iS_1}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_1}-E_{S_i}-\omega_2+i0)(E_{S_2}-E_{S_i}+i0)(E_{S_f}-E_{S_i}-\omega_1+i0)}\Bigg)\\
&\quad\times\frac{P^2_{S_i}}{P_{S_i}+P_{S_1}}\\
&+\Bigg(\frac{\Gamma^{K_1}_{S_fS_i,S_fS_2}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_iS_1,S_2S_1}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_i}-E_{S_1}+\omega_2+i0)(E_{S_i}-E_{S_2}+i0)(E_{S_i}-E_{S_f}-\omega_1+i0)}\\
&\quad+\frac{\Gamma^{K_1}_{S_iS_f,S_2S_f}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_1S_i,S_1S_2}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_1}-E_{S_i}+\omega_2+i0)(E_{S_2}-E_{S_i}+i0)(E_{S_f}-E_{S_i}-\omega_1+i0)}\Bigg)\\
&\quad\times\left(P_{S_2}+\frac{P_{S_i}P_{S_f}}{P_{S_i}+P_{S_1}}\right)\frac{P_{S_i}}{P_{S_2}+P_{S_f}}\quad\Bigg]
\end{aligned}
$$

B.(1).(t) projections

This contribution is of the type $W^{(4)}_{S_f\leftarrow S_i}\,P_{S_i}$

Diagram (b)

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\\
&\Big(\;\sigma^{K_1\,>}_{S_fS_2,S_fS_i}(t-t')\,g^{<}_{S_fS_i}(t'-t)\,g^{<}_{S_1S_i}(t-t_2)\,\langle\hat F_{S_1S_i,S_1S_i}(t_2)\rangle\,\sigma^{K_2\,>}_{S_1S_i,S_1S_2}(t_2-t_1)\,g^{>}_{S_1S_2}(t_1-t)\\
&-\sigma^{K_1\,<}_{S_2S_f,S_iS_f}(t-t')\,g^{>}_{S_iS_f}(t'-t)\,g^{>}_{S_iS_1}(t-t_2)\,\langle\hat F_{S_iS_1,S_iS_1}(t_2)\rangle\,\sigma^{K_2\,<}_{S_iS_1,S_2S_1}(t_2-t_1)\,g^{<}_{S_2S_1}(t_1-t)\Big)
\end{aligned}
$$

Diagram (i)

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\\
&\Big(\;\sigma^{K_1>}_{S_fS_2,S_fS_i}(t-t')\,g^{<}_{S_fS_i}(t'-t)\,B^{<}_{S_2S_i}(t-t_1)\,\sigma^{K_2<}_{S_2S_1,S_iS_1}(t_1-t_2)\,g^{>}_{S_iS_1}(t_2-t_1)\\
&+\sigma^{K_1<}_{S_2S_f,S_iS_f}(t-t')\,g^{>}_{S_iS_f}(t'-t)\,B^{>}_{S_iS_2}(t-t_1)\,\sigma^{K_2>}_{S_1S_2,S_1S_i}(t_1-t_2)\,g^{<}_{S_1S_i}(t_2-t_1)\\
&+\sigma^{K_1>}_{S_fS_2,S_fS_i}(t-t')\,g^{<}_{S_fS_i}(t'-t)\,B^{<}_{S_2S_i}(t-t_2)\,\sigma^{K_2>}_{S_1S_i,S_1S_2}(t_2-t_1)\,g^{<}_{S_1S_2}(t_1-t_2)\\
&+\sigma^{K_1<}_{S_2S_f,S_iS_f}(t-t')\,g^{>}_{S_iS_f}(t'-t)\,B^{>}_{S_iS_2}(t-t_2)\,\sigma^{K_2<}_{S_iS_1,S_2S_1}(t_2-t_1)\,g^{>}_{S_2S_1}(t_1-t_2)\Big)
\end{aligned}
$$

Resulting zero-order expression

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\\
&\Big[\quad\Big(\frac{\Gamma^{K_1}_{S_fS_2,S_fS_i}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_2S_1,S_iS_1}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_1}-E_{S_i}-\omega_2+i0)(E_{S_2}-E_{S_i}+i0)(E_{S_i}-E_{S_f}-\omega_1+i0)}\\
&\quad+\frac{\Gamma^{K_1}_{S_2S_f,S_iS_f}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_1S_2,S_1S_i}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_i}-E_{S_1}-\omega_2+i0)(E_{S_i}-E_{S_2}+i0)(E_{S_f}-E_{S_i}-\omega_1+i0)}\Big)\\
&\quad\times\frac{P^3_{S_i}}{(P_{S_i}+P_{S_f})(P_{S_i}+P_{S_1})}\\
&+\quad\Big(\frac{\Gamma^{K_1}_{S_fS_2,S_fS_i}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_1S_i,S_1S_2}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_1}-E_{S_i}+\omega_2+i0)(E_{S_2}-E_{S_i}+i0)(E_{S_i}-E_{S_f}-\omega_1+i0)}\\
&\quad+\frac{\Gamma^{K_1}_{S_2S_f,S_iS_f}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_iS_1,S_2S_1}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_i}-E_{S_1}+\omega_2+i0)(E_{S_i}-E_{S_2}+i0)(E_{S_f}-E_{S_i}-\omega_1+i0)}\Big)\\
&\quad\times\frac{P^2_{S_i}}{P_{S_i}+P_{S_f}}\quad\Big]
\end{aligned}
$$

B.(2).(t) projections

This contribution is of the type $W^{(4)}_{S_i\leftarrow S_i}\,P_{S_i}$

Diagram (b)

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2}\ \sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\\
&\Big(\ \sigma^{K_1<}_{S_fS_2,S_iS_2}(t-t')\,g^{>}_{S_iS_2}(t'-t)\,g^{>}_{S_iS_1}(t-t_2)\,\langle\hat{F}_{S_iS_1,S_iS_1}(t_2)\rangle\,\sigma^{K_2<}_{S_iS_1,S_fS_1}(t_2-t_1)\,g^{<}_{S_fS_1}(t_1-t)\\
&-\sigma^{K_1>}_{S_2S_f,S_2S_i}(t-t')\,g^{<}_{S_2S_i}(t'-t)\,g^{<}_{S_1S_i}(t-t_2)\,\langle\hat{F}_{S_1S_i,S_1S_i}(t_2)\rangle\,\sigma^{K_2>}_{S_1S_i,S_1S_f}(t_2-t_1)\,g^{>}_{S_1S_f}(t_1-t)\Big)
\end{aligned}
$$

Diagram (i)

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2}\ \sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\\
&\Big(\ \sigma^{K_1<}_{S_fS_2,S_iS_2}(t-t')\,g^{>}_{S_iS_2}(t'-t)\,B^{>}_{S_iS_f}(t-t_2)\,\sigma^{K_2<}_{S_iS_1,S_fS_1}(t_2-t_1)\,g^{>}_{S_fS_1}(t_1-t_2)\\
&+\sigma^{K_1>}_{S_2S_f,S_2S_i}(t-t')\,g^{<}_{S_2S_i}(t'-t)\,B^{<}_{S_fS_i}(t-t_2)\,\sigma^{K_2>}_{S_1S_i,S_1S_f}(t_2-t_1)\,g^{<}_{S_1S_f}(t_1-t_2)\\
&+\sigma^{K_1<}_{S_fS_2,S_iS_2}(t-t')\,g^{>}_{S_iS_2}(t'-t)\,B^{>}_{S_iS_f}(t-t_1)\,\sigma^{K_2>}_{S_1S_f,S_1S_i}(t_1-t_2)\,g^{<}_{S_1S_i}(t_2-t_1)\\
&+\sigma^{K_1>}_{S_2S_f,S_2S_i}(t-t')\,g^{<}_{S_2S_i}(t'-t)\,B^{<}_{S_fS_i}(t-t_1)\,\sigma^{K_2<}_{S_fS_1,S_iS_1}(t_1-t_2)\,g^{>}_{S_iS_1}(t_2-t_1)\Big)
\end{aligned}
$$

Resulting zero-order expression

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\\
\Bigg[\quad&\Bigg(\frac{\Gamma^{K_1}_{S_fS_2,S_iS_2}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_1S_f,S_1S_i}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_i}-E_{S_1}-\omega_2+i0)(E_{S_i}-E_{S_f}+i0)(E_{S_2}-E_{S_i}-\omega_1+i0)}\\
&+\frac{\Gamma^{K_1}_{S_2S_f,S_2S_i}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_fS_1,S_iS_1}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_1}-E_{S_i}-\omega_2+i0)(E_{S_f}-E_{S_i}+i0)(E_{S_i}-E_{S_2}-\omega_1+i0)}\Bigg)\\
&\times\frac{P^3_{S_i}}{(P_{S_i}+P_{S_1})(P_{S_i}+P_{S_2})}\\
&+\Bigg(\frac{\Gamma^{K_1}_{S_fS_2,S_iS_2}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_iS_1,S_fS_1}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_i}-E_{S_1}+\omega_2+i0)(E_{S_i}-E_{S_f}+i0)(E_{S_2}-E_{S_i}-\omega_1+i0)}\\
&+\frac{\Gamma^{K_1}_{S_2S_f,S_2S_i}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_1S_i,S_1S_f}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_1}-E_{S_i}+\omega_2+i0)(E_{S_f}-E_{S_i}+i0)(E_{S_i}-E_{S_2}-\omega_1+i0)}\Bigg)\\
&\times\frac{P^2_{S_i}}{P_{S_i}+P_{S_2}}\quad\Bigg]
\end{aligned}
$$

C.(0).(s) projections

This contribution is of the type $W^{(4)}_{S_i\leftarrow S_i}\,P_{S_i}$

Diagram (c)

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2,S_3}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\int_{-\infty}^{t_2}dt'\\
&\Bigg(\ \sigma^{K_1>}_{S_1S_i,S_3S_i}(t-t')\,g^{<}_{S_3S_i}(t'-t_2)\,g^{<}_{S_2S_1}(t_2-t_1)\,\sigma^{K_2>}_{S_2S_1,S_2S_3}(t_1-t_2)\,g^{<}_{S_1S_i}(t_2-t)\langle\hat{F}_{S_1S_i,S_1S_i}(t)\rangle\\
&-\sigma^{K_1<}_{S_iS_1,S_iS_3}(t-t')\,g^{>}_{S_iS_3}(t'-t_2)\,g^{>}_{S_1S_2}(t_2-t_1)\,\sigma^{K_2<}_{S_1S_2,S_3S_2}(t_1-t_2)\,g^{>}_{S_iS_1}(t_2-t)\langle\hat{F}_{S_iS_1,S_iS_1}(t)\rangle\Bigg)
\end{aligned}
$$

Diagram (d)

$$2\,\mathrm{Im}\sum_{S_1,S_2,S_3}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\int_{-\infty}^{t_2}dt'$$
$$\Big(\ \sigma^{K_1>}_{S_1S_i,S_3S_i}(t-t')\,g^{<}_{S_3S_i}(t'-t_2)\,\sigma^{K_2<}_{S_3S_2,S_1S_2}(t_2-t_1)\,g^{>}_{S_1S_2}(t_1-t_2)\,g^{<}_{S_1S_i}(t_2-t)\langle\hat F_{S_1S_i,S_1S_i}(t)\rangle$$
$$-\,\sigma^{K_1<}_{S_iS_1,S_iS_3}(t-t')\,g^{>}_{S_iS_3}(t'-t_2)\,\sigma^{K_2>}_{S_2S_3,S_2S_1}(t_2-t_1)\,g^{<}_{S_2S_1}(t_1-t_2)\,g^{>}_{S_iS_1}(t_2-t)\langle\hat F_{S_iS_1,S_iS_1}(t)\rangle\Big)$$

Diagram (g)

$$2\,\mathrm{Im}\sum_{S_1,S_2,S_3}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\int_{-\infty}^{t_2}dt'$$
$$\Big(\sigma^{K_1<}_{S_iS_1,S_iS_3}(t-t')\,g^{>}_{S_iS_3}(t'-t_2)\,\sigma^{K_2>}_{S_2S_3,S_2S_1}(t_2-t_1)\,g^{>}_{S_2S_1}(t_1-t)\,B^{<}_{S_2S_i}(t-t_2)$$
$$+\,\sigma^{K_1>}_{S_1S_i,S_3S_i}(t-t')\,g^{<}_{S_3S_i}(t'-t_2)\,\sigma^{K_2<}_{S_3S_2,S_1S_2}(t_2-t_1)\,g^{<}_{S_1S_2}(t_1-t)\,B^{>}_{S_iS_2}(t-t_2)\Big)$$

Diagram (h)

$$2\,\mathrm{Im}\sum_{S_1,S_2,S_3}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\int_{-\infty}^{t_2}dt'$$
$$\Big(\ \sigma^{K_1>}_{S_1S_i,S_3S_i}(t-t')\,g^{<}_{S_3S_i}(t'-t_2)\,D^{<}_{S_2S_i}(t_2-t)\,g^{>}_{S_2S_1}(t-t_1)\,\sigma^{K_2>}_{S_2S_1,S_2S_3}(t_1-t_2)$$
$$+\,\sigma^{K_1<}_{S_iS_1,S_iS_3}(t-t')\,g^{>}_{S_iS_3}(t'-t_2)\,D^{>}_{S_iS_2}(t_2-t)\,g^{<}_{S_1S_2}(t-t_1)\,\sigma^{K_2<}_{S_1S_2,S_3S_2}(t_1-t_2)\Big)$$

Resulting zero-order expression

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2,S_3}\sum_{K_1,K_2\in\{L,R\}}\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\,\frac{P_{S_i}^2}{P_{S_i}+P_{S_3}}\\
&\Big[\frac{\Gamma^{K_1}_{S_iS_1,S_iS_3}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_2S_3,S_2S_1}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_3}-E_{S_i}-\omega_1+i0)(E_{S_2}-E_{S_i}-\omega_1+\omega_2+i0)(E_{S_1}-E_{S_i}-\omega_1+i0)}\\
&+\frac{\Gamma^{K_1}_{S_iS_1,S_iS_3}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_1S_2,S_3S_2}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_3}-E_{S_i}-\omega_1+i0)(E_{S_2}-E_{S_i}-\omega_1-\omega_2+i0)(E_{S_1}-E_{S_i}-\omega_1+i0)}\\
&+\frac{\Gamma^{K_1}_{S_1S_i,S_3S_i}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_2S_1,S_2S_3}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_i}-E_{S_3}-\omega_1+i0)(E_{S_i}-E_{S_2}-\omega_1-\omega_2+i0)(E_{S_i}-E_{S_1}-\omega_1+i0)}\\
&+\frac{\Gamma^{K_1}_{S_1S_i,S_3S_i}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_3S_2,S_1S_2}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_i}-E_{S_3}-\omega_1+i0)(E_{S_i}-E_{S_2}-\omega_1+\omega_2+i0)(E_{S_i}-E_{S_1}-\omega_1+i0)}\Big]
\end{aligned}
$$

C.(1).(s) projections

This contribution is of the type $W^{(4)}_{S_f\leftarrow S_i}\,P_{S_i}$

Diagram (c)

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\int_{-\infty}^{t_2}dt'\\
&\Big(\,\sigma^{K_1\,<}_{S_iS_f,S_iS_1}(t-t')\,g^{>}_{S_iS_1}(t'-t_2)\,g^{>}_{S_fS_2}(t_2-t_1)\,\sigma^{K_2\,<}_{S_fS_2,S_1S_2}(t_1-t_2)\,g^{>}_{S_iS_f}(t_2-t)\langle\hat{F}_{S_iS_f,S_iS_f}(t)\rangle\\
&-\sigma^{K_1\,>}_{S_fS_i,S_1S_i}(t-t')\,g^{<}_{S_1S_i}(t'-t_2)\,g^{<}_{S_2S_f}(t_2-t_1)\,\sigma^{K_2\,>}_{S_2S_f,S_2S_1}(t_1-t_2)\,g^{<}_{S_fS_i}(t_2-t)\langle\hat{F}_{S_fS_i,S_fS_i}(t)\rangle\Big)
\end{aligned}
$$

Diagram (d)

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\int_{-\infty}^{t_1}dt'\\
&\Big(\,\sigma^{K_1\,<}_{S_iS_f,S_iS_1}(t-t')\,g^{>}_{S_iS_1}(t'-t_2)\,\sigma^{K_2\,>}_{S_2S_1,S_2S_f}(t_2-t_1)\,g^{<}_{S_2S_f}(t_1-t_2)\,g^{>}_{S_iS_f}(t_2-t)\,\langle\hat{F}_{S_iS_f,S_iS_f}(t)\rangle\\
&-\sigma^{K_1\,>}_{S_fS_i,S_1S_i}(t-t')\,g^{<}_{S_1S_i}(t'-t_2)\,\sigma^{K_2\,<}_{S_1S_2,S_fS_2}(t_2-t_1)\,g^{>}_{S_fS_2}(t_1-t_2)\,g^{<}_{S_fS_i}(t_2-t)\,\langle\hat{F}_{S_fS_i,S_fS_i}(t)\rangle\Big)
\end{aligned}
$$

Diagram (g)

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\int_{-\infty}^{t_2}dt'\\
&\Big(\,\sigma^{K_1>}_{S_fS_i,S_1S_i}(t-t')\,g^{<}_{S_1S_i}(t'-t_2)\,\sigma^{K_2<}_{S_1S_2,S_fS_2}(t_2-t_1)\,g^{<}_{S_fS_2}(t_1-t)\,B^{>}_{S_iS_2}(t-t_2)\\
&+\sigma^{K_1<}_{S_iS_f,S_iS_1}(t-t')\,g^{>}_{S_iS_1}(t'-t_2)\,\sigma^{K_2>}_{S_2S_1,S_2S_f}(t_2-t_1)\,g^{>}_{S_2S_f}(t_1-t)\,B^{<}_{S_2S_i}(t-t_2)\Big)
\end{aligned}
$$

Diagram (h)

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt_1\int_{-\infty}^{t_1}dt_2\int_{-\infty}^{t_2}dt'\\
&\Big(\,\sigma^{K_1>}_{S_fS_i,S_1S_i}(t-t')\,g^{<}_{S_1S_i}(t'-t_2)\,D^{<}_{S_2S_i}(t_2-t)\,g^{>}_{S_2S_f}(t-t_1)\,\sigma^{K_2>}_{S_2S_f,S_2S_1}(t_1-t_2)\\
&+\sigma^{K_1<}_{S_iS_f,S_iS_1}(t-t')\,g^{>}_{S_iS_1}(t'-t_2)\,D^{>}_{S_iS_2}(t_2-t)\,g^{<}_{S_fS_2}(t-t_1)\,\sigma^{K_2<}_{S_fS_2,S_1S_2}(t_1-t_2)\Big)
\end{aligned}
$$

Resulting zero-order expression

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\,\frac{P^2_{S_i}}{P_{S_i}+P_{S_1}}\\
&\Big[\,\frac{\Gamma^{K_1}_{S_fS_i,S_1S_i}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_2S_f,S_2S_1}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_i}-E_{S_1}-\omega_1+i0)(E_{S_i}-E_{S_2}-\omega_1-\omega_2+i0)(E_{S_i}-E_{S_f}-\omega_1+i0)}\\
&+\frac{\Gamma^{K_1}_{S_fS_i,S_1S_i}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_1S_2,S_fS_2}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_i}-E_{S_1}-\omega_1+i0)(E_{S_i}-E_{S_2}-\omega_1+\omega_2+i0)(E_{S_i}-E_{S_f}-\omega_1+i0)}\\
&+\frac{\Gamma^{K_1}_{S_iS_f,S_iS_1}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_2S_1,S_2S_f}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_1}-E_{S_i}-\omega_1+i0)(E_{S_2}-E_{S_i}-\omega_1+\omega_2+i0)(E_{S_f}-E_{S_i}-\omega_1+i0)}\\
&+\frac{\Gamma^{K_1}_{S_iS_f,S_iS_1}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_fS_2,S_1S_2}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_1}-E_{S_i}-\omega_1+i0)(E_{S_2}-E_{S_i}-\omega_1-\omega_2+i0)(E_{S_f}-E_{S_i}-\omega_1+i0)}\Big]
\end{aligned}
$$

C.(1).(t) projections

This contribution is of the type $W^{(4)}_{S_f\leftarrow S_i}\,P_{S_i}$

Diagram (b)

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt_2\int_{-\infty}^{t}dt_1\\
&\Big(\;\sigma^{K_1\,>}_{S_2S_f,S_2S_1}(t-t')\,g^{<}_{S_2S_1}(t'-t)\,g^{>}_{S_iS_1}(t-t_2)\,\langle\hat{F}_{S_iS_1,S_iS_1}(t_2)\rangle\,\sigma^{K_2\,<}_{S_iS_1,S_iS_f}(t_2-t_1)\,g^{>}_{S_iS_f}(t_1-t)\\
&-\sigma^{K_1\,<}_{S_fS_2,S_1S_2}(t-t')\,g^{>}_{S_1S_2}(t'-t)\,g^{<}_{S_1S_i}(t-t_2)\,\langle\hat{F}_{S_1S_i,S_1S_i}(t_2)\rangle\,\sigma^{K_2\,>}_{S_1S_i,S_fS_i}(t_2-t_1)\,g^{<}_{S_fS_i}(t_1-t)\\
&+\sigma^{K_1\,<}_{S_fS_2,S_1S_2}(t-t')\,g^{>}_{S_1S_2}(t'-t)\,g^{>}_{S_iS_f}(t-t_1)\,\langle\hat{F}_{S_iS_f,S_iS_f}(t_1)\rangle\,\sigma^{K_2\,<}_{S_iS_f,S_iS_1}(t_1-t_2)\,g^{>}_{S_iS_1}(t_2-t)\\
&-\sigma^{K_1\,>}_{S_2S_f,S_2S_1}(t-t')\,g^{<}_{S_2S_1}(t'-t)\,g^{<}_{S_fS_i}(t-t_1)\,\langle\hat{F}_{S_fS_i,S_fS_i}(t_1)\rangle\,\sigma^{K_2\,>}_{S_fS_i,S_1S_i}(t_1-t_2)\,g^{<}_{S_1S_i}(t_2-t)\Big)
\end{aligned}
$$

Diagram (g)

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt_2\int_{-\infty}^{t}dt_1\\
&\Big(\;\sigma^{K_1\,<}_{S_fS_2,S_1S_2}(t-t')\,g^{<}_{S_1S_2}(t'-t_2)\,\sigma^{K_2\,>}_{S_1S_i,S_fS_i}(t_2-t_1)\,g^{<}_{S_fS_i}(t_1-t)\,B^{<}_{S_2S_i}(t-t_2)\\
&+\sigma^{K_1\,>}_{S_2S_f,S_2S_1}(t-t')\,g^{>}_{S_2S_1}(t'-t_2)\,\sigma^{K_2\,<}_{S_iS_1,S_iS_f}(t_2-t_1)\,g^{>}_{S_iS_f}(t_1-t)\,B^{>}_{S_iS_2}(t-t_2)\Big)
\end{aligned}
$$

Diagram (h)

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt_2\int_{-\infty}^{t}dt_1\\
&\Big(\;\sigma^{K_1\,<}_{S_fS_2,S_1S_2}(t-t')\,g^{<}_{S_1S_2}(t'-t_2)\,D^{>}_{S_iS_2}(t_2-t)\,g^{>}_{S_iS_f}(t-t_1)\,\sigma^{K_2\,<}_{S_iS_f,S_iS_1}(t_1-t_2)\\
&+\sigma^{K_1\,>}_{S_2S_f,S_2S_1}(t-t')\,g^{>}_{S_2S_1}(t'-t_2)\,D^{<}_{S_2S_i}(t_2-t)\,g^{<}_{S_fS_i}(t-t_1)\,\sigma^{K_2\,>}_{S_fS_i,S_1S_i}(t_1-t_2)\Big)
\end{aligned}
$$

Resulting zero-order expression

$$
\begin{aligned}
&-2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}\\
&\Bigg[\quad\Bigg(\frac{\Gamma^{K_1}_{S_fS_2,S_1S_2}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_1S_i,S_fS_i}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_1}-E_{S_i}+\omega_2+i0)(E_{S_2}-E_{S_i}-\omega_1+\omega_2+i0)(E_{S_i}-E_{S_f}-\omega_2+i0)}\\
&\quad+\frac{\Gamma^{K_1}_{S_2S_f,S_2S_1}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_iS_1,S_iS_f}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_i}-E_{S_1}+\omega_2+i0)(E_{S_i}-E_{S_2}-\omega_1+\omega_2+i0)(E_{S_f}-E_{S_i}-\omega_2+i0)}\Bigg)\\
&\quad\times\frac{P^2_{S_i}}{P_{S_i}+P_{S_f}}\\
&\quad+\Bigg(\frac{\Gamma^{K_1}_{S_fS_2,S_1S_2}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_iS_f,S_iS_1}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_1}-E_{S_i}-\omega_2+i0)(E_{S_2}-E_{S_i}-\omega_1-\omega_2+i0)(E_{S_i}-E_{S_f}+\omega_2+i0)}\\
&\quad+\frac{\Gamma^{K_1}_{S_2S_f,S_2S_1}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_fS_i,S_1S_i}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_i}-E_{S_1}-\omega_2+i0)(E_{S_i}-E_{S_2}-\omega_1-\omega_2+i0)(E_{S_f}-E_{S_i}+\omega_2+i0)}\Bigg)\\
&\quad\times\left(\frac{P_{S_2}}{P_{S_i}+P_{S_f}}+\frac{P_{S_1}}{P_{S_i}+P_{S_1}}\right)\frac{P^2_{S_i}}{P_{S_1}+P_{S_2}}\quad\Bigg]
\end{aligned}
$$

C.(2).(t) projections

This contribution is of the type $W^{(4)}_{S_f\leftarrow S_i}\,P_{S_i}$

Diagram (b)

$$
\begin{aligned}
&2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt_2\int_{-\infty}^{t}dt_1\\
&\Big(\ \sigma^{K_1\,>}_{S_fS_2,S_fS_1}(t-t')\,g^{<}_{S_fS_1}(t'-t)\,g^{<}_{S_2S_i}(t-t_1)\,\langle\hat F_{S_2S_i,S_2S_i}(t_1)\rangle\,\sigma^{K_2\,>}_{S_2S_i,S_1S_i}(t_1-t_2)\,g^{<}_{S_1S_i}(t_2-t)\\
&-\sigma^{K_1\,<}_{S_2S_f,S_1S_f}(t-t')\,g^{>}_{S_1S_f}(t'-t)\,g^{>}_{S_iS_2}(t-t_1)\,\langle\hat F_{S_iS_2,S_iS_2}(t_1)\rangle\,\sigma^{K_2\,<}_{S_iS_2,S_iS_1}(t_1-t_2)\,g^{>}_{S_iS_1}(t_2-t)\\
&+\sigma^{K_1\,<}_{S_2S_f,S_1S_f}(t-t')\,g^{>}_{S_1S_f}(t'-t)\,g^{<}_{S_1S_i}(t-t_2)\,\langle\hat F_{S_1S_i,S_1S_i}(t_2)\rangle\,\sigma^{K_2\,>}_{S_1S_i,S_2S_i}(t_2-t_1)\,g^{<}_{S_2S_i}(t_1-t)\\
&-\sigma^{K_1\,>}_{S_fS_2,S_fS_1}(t-t')\,g^{<}_{S_fS_1}(t'-t)\,g^{>}_{S_iS_1}(t-t_2)\,\langle\hat F_{S_iS_1,S_iS_1}(t_2)\rangle\,\sigma^{K_2\,<}_{S_iS_1,S_iS_2}(t_2-t_1)\,g^{>}_{S_iS_2}(t_1-t)\Big)
\end{aligned}
$$

Diagram (g)

$$2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt_2\int_{-\infty}^{t}dt_1$$
$$\Big(\ \sigma^{K_1\,>}_{S_fS_2,S_fS_1}(t-t')\,g^{>}_{S_fS_1}(t'-t_2)\,\sigma^{K_2\,<}_{S_iS_1,S_iS_2}(t_2-t_1)\,g^{>}_{S_iS_2}(t_1-t)\,B^{>}_{S_iS_f}(t-t_2)$$
$$+\,\sigma^{K_1\,<}_{S_2S_f,S_1S_f}(t-t')\,g^{<}_{S_1S_f}(t'-t_2)\,\sigma^{K_2\,>}_{S_1S_i,S_2S_i}(t_2-t_1)\,g^{<}_{S_2S_i}(t_1-t)\,B^{<}_{S_fS_i}(t-t_2)\Big)$$

Diagram (h)

$$2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int_{-\infty}^{t}dt'\int_{-\infty}^{t'}dt_2\int_{-\infty}^{t}dt_1$$
$$\Big(\ \sigma^{K_1\,>}_{S_fS_2,S_fS_1}(t-t')\,g^{>}_{S_fS_1}(t'-t_2)\,D^{<}_{S_fS_i}(t_2-t)\,g^{<}_{S_2S_i}(t-t_1)\,\sigma^{K_2\,>}_{S_2S_i,S_1S_i}(t_1-t_2)$$
$$+\,\sigma^{K_1\,<}_{S_2S_f,S_1S_f}(t-t')\,g^{<}_{S_1S_f}(t'-t_2)\,D^{>}_{S_iS_f}(t_2-t)\,g^{>}_{S_iS_2}(t-t_1)\,\sigma^{K_2\,<}_{S_iS_2,S_iS_1}(t_1-t_2)\Big)$$

Resulting zero-order expression

$$2\,\mathrm{Im}\sum_{S_1,S_2}\sum_{K_1,K_2\in\{L,R\}}\int\frac{d\omega_1}{2\pi}\int\frac{d\omega_2}{2\pi}$$
$$\Big[\quad\Big(\frac{\Gamma^{K_1}_{S_fS_2,S_fS_1}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_iS_1,S_iS_2}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_i}-E_{S_1}+\omega_2+i0)(E_{S_i}-E_{S_f}-\omega_1+\omega_2+i0)(E_{S_2}-E_{S_i}-\omega_2+i0)}$$
$$+\,\frac{\Gamma^{K_1}_{S_2S_f,S_1S_f}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_1S_i,S_2S_i}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_1}-E_{S_i}+\omega_2+i0)(E_{S_f}-E_{S_i}-\omega_1+\omega_2+i0)(E_{S_i}-E_{S_2}-\omega_2+i0)}\Big)$$
$$\times\,\frac{P_{S_i}^2}{P_{S_i}+P_{S_2}}$$
$$+\,\Big(\frac{\Gamma^{K_1}_{S_fS_2,S_fS_1}(\omega_1)[1-f_{K_1}(\omega_1)]\Gamma^{K_2}_{S_2S_i,S_1S_i}(\omega_2)[1-f_{K_2}(\omega_2)]}{(E_{S_i}-E_{S_1}-\omega_2+i0)(E_{S_i}-E_{S_f}-\omega_1-\omega_2+i0)(E_{S_2}-E_{S_i}+\omega_2+i0)}$$
$$+\,\frac{\Gamma^{K_1}_{S_2S_f,S_1S_f}(\omega_1)f_{K_1}(\omega_1)\Gamma^{K_2}_{S_iS_2,S_iS_1}(\omega_2)f_{K_2}(\omega_2)}{(E_{S_1}-E_{S_i}-\omega_2+i0)(E_{S_f}-E_{S_i}-\omega_1-\omega_2+i0)(E_{S_i}-E_{S_2}+\omega_2+i0)}\Big)$$
$$\times\Big(\frac{P_{S_1}}{P_{S_i}+P_{S_1}}+\frac{P_{S_f}}{P_{S_i}+P_{S_2}}\Big)\frac{P_{S_i}^2}{P_{S_1}+P_{S_f}}\quad\Big]$$

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