Electronic Supporting Information

Analytical expressions for spring constants of capillary bridges and snap-in forces of hydrophobic surfaces

Veikko Sariola

Tampere University, Faculty of Medicine and Health Technology, Korkeakoulunkatu 3, 33720 Tampere, Finland

Number of pages: 14

Manipulating Kenmotsu's parameterization (2-3 in the paper) to the form given by equations 4-5 in the paper:

$r_ := \frac{1}{2\,H}\,\mathrm{sqrt}\left(1 + B^2 + 2\,B\cdot\sin(2\cdot H\cdot s)\right)$

$$r_ := \frac{\sqrt{B^2 + 1 + 2\,B\,\sin(2\,H\,s)}}{2\,H} \tag{1}$$

$simplify\left(applyrule\left(\cos(2\cdot x :: anything) = 2\cdot\cos(x)^2 - 1, subs\left(\sin = \cos, H = \frac{1}{R}, s = R\cdot s, r_\right)\right)\right)$

$$\frac{R\sqrt{4\,B\,\cos(s)^2 + (B-1)^2}}{2} \tag{2}$$

which is exactly equation 4 in the paper.

$z_ := Int\left(\frac{(1 + B\cdot\sin(2\cdot H\cdot t))}{\mathrm{sqrt}\left(1 + B^2 + 2\,B\cdot\sin(2\cdot H\cdot t)\right)}, t = 0\,..s\right)$

$$z_ := \int_0^s \frac{1 + B\,\sin(2\,H\,t)}{\sqrt{1 + B^2 + 2\,B\,\sin(2\,H\,t)}}\,dt \tag{3}$$

$subs\left(s = R\cdot s, IntegrationTools[Change]\left(subs\left(\sin = \cos, H = \frac{1}{R}, z_\right), t = R\cdot u, u\right)\right)$

$$R\left(\int_0^s \frac{1 + B\,\cos(2\,u)}{\sqrt{1 + B^2 + 2\,B\,\cos(2\,u)}}\,du\right) \tag{4}$$

$simplify\left(applyrule\left(\cos(2\cdot x :: anything) = 2\cdot\cos(x)^2 - 1, \mathbf{(4)}\right)\right)$

$$R\left(\int_0^s \frac{2\,B\,\cos(u)^2 - B + 1}{\sqrt{4\,B\,\cos(u)^2 + (B-1)^2}}\,du\right) \tag{5}$$

which are exactly the equation 5 in the paper.

Equation 4 in the paper:

$r := \frac{R}{2}\,\mathrm{sqrt}\left((1 - B)^2 + 4\cdot B\cdot\cos(s)^2\right)$

$$r := \frac{R\sqrt{(-B+1)^2 + 4\,B\,\cos(s)^2}}{2} \tag{6}$$

Equation 5 in the paper:

$z := Int\left(R\cdot\frac{\left((1 - B) + 2\cdot B\cdot\cos(t)^2\right)}{\mathrm{sqrt}\left((1 - B)^2 + 4\cdot B\cdot\cos(t)^2\right)}, t = 0\,..s\right)$

$$z := \int_0^s \frac{R\left(1 - B + 2\,B\,\cos(t)^2\right)}{\sqrt{(-B+1)^2 + 4\,B\,\cos(t)^2}}\,dt \tag{7}$$

Equation 6 in the paper:

$$assume\left(-\frac{\mathrm{Pi}}{2} < alpha1 < \frac{\mathrm{Pi}}{2}, -\frac{\mathrm{Pi}}{2} < alpha2 < \frac{\mathrm{Pi}}{2}, alpha1 < alpha2\right) : interface(showassumed = 0) :$$

$$h := combine(subs(s = alpha2, z) - subs(s = alpha1, z), 'range')$$

$$h := \int_{\alpha l}^{\alpha 2} \frac{R\left(1 - B + 2\,B\cos(t)^2\right)}{\sqrt{(-B+1)^2 + 4\,B\cos(t)^2}}\,dt \tag{8}$$

$$V := \mathrm{Pi}\cdot Int\left(r^2 \cdot diff(z, s), s = alpha1 \,..\, alpha2\right)$$

$$V := \pi\left(\int_{\alpha l}^{\alpha 2} \frac{R^3\sqrt{(-B+1)^2 + 4\,B\cos(s)^2}\,\left(1 - B + 2\,B\cos(s)^2\right)}{4}\,ds\right) \tag{9}$$

$$r1 := subs(s = alpha1, r)$$

$$r1 := \frac{R\sqrt{(-B+1)^2 + 4\,B\cos(\alpha l)^2}}{2} \tag{10}$$

$$r2 := subs(s = alpha2, r)$$

$$r2 := \frac{R\sqrt{(-B+1)^2 + 4\,B\cos(\alpha 2)^2}}{2} \tag{11}$$

Equation 7 in the paper:

$$F := \mathrm{Pi}\cdot r1^2 \cdot p - 2\cdot\mathrm{Pi}\cdot\mathrm{gamma}\cdot r1\cdot\sin(theta1)$$

$$F := \frac{\pi R^2\left((-B+1)^2 + 4\,B\cos(\alpha l)^2\right)p}{4} - \pi\gamma R\sqrt{(-B+1)^2 + 4\,B\cos(\alpha l)^2}\,\sin(\theta l) \tag{12}$$

The previous was simplified to arrive at equation 8 in the paper (since we will be using this with B = 1, the assumption is valid):

$$simplify\left(subs\left(\left\{p = \frac{2\cdot\mathrm{gamma}}{R}, \sin(theta1) = \frac{1}{\mathrm{sqrt}\left(1 + subs\left(s = alpha1, \frac{diff(r, s)^2}{diff(z, s)^2}\right)\right)}\right\}, (12)\right)\right)$$

$$\text{assuming } 1 - B + 2\,B\cos(\alpha l)^2 \geq 0$$

$$\frac{\pi R\gamma\left(B^2 - 1\right)}{2} \tag{13}$$

Equation 9 in the paper:

$$value(simplify(subs(B = 1, \{r, z\}))) \text{ assuming } -\frac{\mathrm{Pi}}{2} < s < \frac{\mathrm{Pi}}{2}$$

$$\{R\cos(s), R\sin(s)\} \tag{14}$$

Equation 10 in the paper:

$$alpha1 = \frac{\mathrm{Pi}}{2} - theta1$$

$$\alpha l = \frac{\pi}{2} - \theta l \tag{15}$$

Equation 11 in the paper:

$$alpha2 = theta2 - \frac{\mathrm{Pi}}{2}$$

$$\alpha 2 = \theta 2 - \frac{\pi}{2} \tag{16}$$

Equation 12 in the paper (with X standing for dB/dh, which we'll replace shortly)

$k := -diff((13), B) \cdot X$

$$k := -\pi R \gamma B X \tag{17}$$

Equation 14 in the paper (**case III**):

$with(VectorCalculus)$:

$with(LinearAlgebra)$:

$simplify(value(subs(\{B = 1, R = 1\}, Determinant(Jacobian([V, r1, r2], [R, alpha1, alpha2])))))$

$$\left(-\sin(\alpha 1)\cos(\alpha 2)^2 + \sin(\alpha 2)\cos(\alpha 1)^2 + 2\sin(\alpha 1) - 2\sin(\alpha 2)\right)\pi \tag{18}$$

$simplify(subs(\{(15), (16)\}, (18)))$

$$\pi\left(\cos(\theta 2) + \cos(\theta 1)\right)\left(\cos(\theta 2)\cos(\theta 1) + 1\right) \tag{19}$$

Equation 15 in the paper (the factors in front of the logarithms should still be collected):

$factor(simplify(subs(\{(15), (16)\}, simplify(value(subs(\{B = 1, R = 1\}, Determinant(Jacobian([h, V, r1, r2], [B, R, alpha1, alpha2])))))))))$

$$\frac{1}{2}\left(\pi\left(\cos(\theta 2) + \cos(\theta 1)\right)\left(\ln(1 + \cos(\theta 1))\cos(\theta 1)\cos(\theta 2) - \ln(1 - \cos(\theta 2))\cos(\theta 1)\cos(\theta 2) - \ln(\sin(\theta 1))\cos(\theta 1)\cos(\theta 2) + \ln(\sin(\theta 2))\cos(\theta 1)\cos(\theta 2) + \ln(1 + \cos(\theta 1)) - \ln(1 - \cos(\theta 2)) - \ln(\sin(\theta 1)) + \ln(\sin(\theta 2)) - \cos(\theta 1) - \cos(\theta 2)\right)\right) \tag{20}$$

Equation 16 (only minor rearrangement needed to arrive at the equation in the paper):

$$collect\left(\frac{2 \cdot \mathrm{Pi} \cdot \gamma}{simplify\left(subs\left(B = 1, R = 1, X = \frac{(19)}{(20)}, (17)\right)\right)}, \ln\right)$$

$$-\ln(1 + \cos(\theta 1)) - \frac{\left(-\cos(\theta 2)\cos(\theta 1) - 1\right)\ln(1 - \cos(\theta 2))}{\cos(\theta 2)\cos(\theta 1) + 1} - \frac{\left(-\cos(\theta 2)\cos(\theta 1) - 1\right)\ln(\sin(\theta 1))}{\cos(\theta 2)\cos(\theta 1) + 1} - \ln(\sin(\theta 2)) - \frac{-\cos(\theta 2) - \cos(\theta 1)}{\cos(\theta 2)\cos(\theta 1) + 1} \tag{21}$$

Equation 17 in the paper (**case II**):

$$slope1 := subs\left(s = alpha1, \frac{diff(r, s)}{diff(z, s)}\right)$$

$$slope1 := -\frac{2B\cos(\alpha 1)\sin(\alpha 1)}{1 - B + 2B\cos(\alpha 1)^2} \tag{22}$$

$simplify(value(subs(\{B = 1, R = 1\}, Determinant(Jacobian([V, slope1, r2], [R, alpha1, alpha2])))))$

$$-\frac{\left(\sin(\alpha 2)\left(\cos(\alpha 1)^2 + 2\right)\sin(\alpha 1) + \cos(\alpha 2)^2 - 2\right)\pi}{\cos(\alpha 1)^2} \tag{23}$$

$simplify(subs(\{(15), (16)\}, (23)))$

$$-\frac{\pi\left(-1 - \cos(\theta 2)^2 + \left(\cos(\theta 1)^3 - 3\cos(\theta 1)\right)\cos(\theta 2)\right)}{\sin(\theta 1)^2} \tag{24}$$

$factor(simplify(subs(\{(15), (16)\}, simplify(value(subs(\{B = 1, R = 1\}, Determinant(Jacobian([h, V, slope1, r2], [B, R, alpha1, alpha2])))))))))$

$$-\frac{1}{2\sin(\theta 1)^2}\left(\pi\left(\ln(1+\cos(\theta 1))\cos(\theta 1)^3\cos(\theta 2)-\ln(1-\cos(\theta 2))\cos(\theta 1)^3\cos(\theta 2)\right.\right. \tag{25}$$
$$-\ln(\sin(\theta 1))\cos(\theta 1)^3\cos(\theta 2)+\ln(\sin(\theta 2))\cos(\theta 1)^3\cos(\theta 2)-3\ln(1$$
$$+\cos(\theta 1))\cos(\theta 1)\cos(\theta 2)-\ln(1+\cos(\theta 1))\cos(\theta 2)^2+3\ln(1$$
$$-\cos(\theta 2))\cos(\theta 1)\cos(\theta 2)+\ln(1-\cos(\theta 2))\cos(\theta 2)^2$$
$$+3\ln(\sin(\theta 1))\cos(\theta 1)\cos(\theta 2)+\ln(\sin(\theta 1))\cos(\theta 2)^2$$
$$-3\ln(\sin(\theta 2))\cos(\theta 1)\cos(\theta 2)-\ln(\sin(\theta 2))\cos(\theta 2)^2-\cos(\theta 1)^3$$
$$-\cos(\theta 1)^2\cos(\theta 2)-\ln(1+\cos(\theta 1))+\ln(1-\cos(\theta 2))+\ln(\sin(\theta 1))$$
$$\left.-\ln(\sin(\theta 2))+\cos(\theta 1)+\cos(\theta 2)\right)\Bigg)$$

$$collect\left(\frac{2\cdot Pi\cdot\gamma}{simplify\left(subs\left(B=1,R=1,X=\frac{(24)}{(25)},(17)\right)\right)},\ln\right)$$

$$-\ln(1+\cos(\theta 1)) \tag{26}$$
$$-\frac{\left(1+\cos(\theta 2)^2+\left(-\cos(\theta 1)^3+3\cos(\theta 1)\right)\cos(\theta 2)\right)\ln(1-\cos(\theta 2))}{-1-\cos(\theta 2)^2+\left(\cos(\theta 1)^3-3\cos(\theta 1)\right)\cos(\theta 2)}$$
$$-\frac{\left(1+\cos(\theta 2)^2+\left(-\cos(\theta 1)^3+3\cos(\theta 1)\right)\cos(\theta 2)\right)\ln(\sin(\theta 1))}{-1-\cos(\theta 2)^2+\left(\cos(\theta 1)^3-3\cos(\theta 1)\right)\cos(\theta 2)}-\ln(\sin(\theta 2))$$
$$-\frac{-\cos(\theta 1)^3-\cos(\theta 1)^2\cos(\theta 2)+\cos(\theta 1)+\cos(\theta 2)}{-1-\cos(\theta 2)^2+\left(\cos(\theta 1)^3-3\cos(\theta 1)\right)\cos(\theta 2)}$$

The factors in front of logarithms cancel out. The last term still needs some work to get eq. 17 in the paper:

$$factor\left(algsubs\left(\cos(theta1)^2=1-\sin(theta1)^2,\right.\right.$$
$$\left.\left.-\frac{-\cos(\theta 1)^3-\cos(\theta 1)^2\cos(\theta 2)+\cos(\theta 1)+\cos(\theta 2)}{-1-\cos(\theta 2)^2+\left(\cos(\theta 1)^3-3\cos(\theta 1)\right)\cos(\theta 2)}\right)\right)$$
$$\frac{\sin(\theta 1)^2(\cos(\theta 2)+\cos(\theta 1))}{\cos(\theta 1)\cos(\theta 2)\sin(\theta 1)^2+2\cos(\theta 2)\cos(\theta 1)+\cos(\theta 2)^2+1} \tag{27}$$

which can be brought to form of the equation 17 in the paper.

Equation 18 in the paper (**case I**):

$$slope2 := subs\left(s=alpha2,\frac{diff(r,s)}{diff(z,s)}\right)$$
$$slope2 := -\frac{2B\cos(\alpha 2)\sin(\alpha 2)}{1-B+2B\cos(\alpha 2)^2} \tag{28}$$

$$simplify(value(subs(\{B=1,R=1\},Determinant(Jacobian([V,slope1,slope2],[R,alpha1,alpha2])))))$$

$$-\frac{\pi\left(\cos(\alpha 1)^2\sin(\alpha 1)-\cos(\alpha 2)^2\sin(\alpha 2)+2\sin(\alpha 1)-2\sin(\alpha 2)\right)}{\cos(\alpha 1)^2\cos(\alpha 2)^2} \tag{29}$$

simplify(*subs*({**(15)**, **(16)**}, **(29)**))

$$\frac{\pi\left(\cos(\theta 1)^3+\cos(\theta 2)^3-3\cos(\theta 1)-3\cos(\theta 2)\right)}{\sin(\theta 1)^2\sin(\theta 2)^2} \tag{30}$$

factor(*simplify*(*subs*({**(15)**, **(16)**}, *simplify*(*value*(*subs*({$B=1$, $R=1$}, *Determinant*(*Jacobian*([h, V, *slope1*, *slope2*], [B, R, *alpha1*, *alpha2*])))))))))

$$\frac{1}{2\sin(\theta 2)^2\sin(\theta 1)^2}\left(\pi\left(\cos(\theta 2)+\cos(\theta 1)\right)\left(\cos(\theta 1)^2\ln(1+\cos(\theta 1))-\ln(1\right.\right. \tag{31}$$
$$+\cos(\theta 1))\cos(\theta 1)\cos(\theta 2)+\ln(1+\cos(\theta 1))\cos(\theta 2)^2-\cos(\theta 1)^2\ln(1-\cos(\theta 2))$$
$$+\ln(1-\cos(\theta 2))\cos(\theta 1)\cos(\theta 2)-\ln(1-\cos(\theta 2))\cos(\theta 2)^2$$
$$-\cos(\theta 1)^2\ln(\sin(\theta 1))+\ln(\sin(\theta 1))\cos(\theta 1)\cos(\theta 2)-\ln(\sin(\theta 1))\cos(\theta 2)^2$$
$$+\cos(\theta 1)^2\ln(\sin(\theta 2))-\ln(\sin(\theta 2))\cos(\theta 1)\cos(\theta 2)+\ln(\sin(\theta 2))\cos(\theta 2)^2$$
$$-3\ln(1+\cos(\theta 1))+3\ln(1-\cos(\theta 2))+3\ln(\sin(\theta 1))-3\ln(\sin(\theta 2))-\cos(\theta 1)$$
$$\left.\left.-\cos(\theta 2)\right)\right)$$

$$collect\left(\frac{2\cdot Pi\cdot\gamma}{simplify\left(subs\left(B=1, R=1, X=\frac{(30)}{(31)}, (17)\right)\right)}, \ln\right)$$

$$-\ln(1+\cos(\theta 1))-\frac{\left(-\cos(\theta 1)^2+\cos(\theta 2)\cos(\theta 1)-\cos(\theta 2)^2+3\right)\ln(1-\cos(\theta 2))}{\cos(\theta 1)^2-\cos(\theta 2)\cos(\theta 1)+\cos(\theta 2)^2-3} \tag{32}$$
$$-\frac{\left(-\cos(\theta 1)^2+\cos(\theta 2)\cos(\theta 1)-\cos(\theta 2)^2+3\right)\ln(\sin(\theta 1))}{\cos(\theta 1)^2-\cos(\theta 2)\cos(\theta 1)+\cos(\theta 2)^2-3}-\ln(\sin(\theta 2))$$
$$-\frac{-\cos(\theta 2)-\cos(\theta 1)}{\cos(\theta 1)^2-\cos(\theta 2)\cos(\theta 1)+\cos(\theta 2)^2-3}$$

which leads to the equation in the paper.

Finally, we will compare our expressions with given ones by Kusumaatmaja & Lipowsky (2010).

For the **case I**, they gave the equation for $-\frac{2\cdot Pi\cdot gamma}{k}$ as:

$$\frac{(\cos(theta1)+\cos(theta2))^2}{3\cdot\cos(theta1)-\cos(theta1)^3+3\cdot\cos(theta2)-\cos(theta2)^3}+\ln(\csc(theta1)+\cot(theta1))$$
$$+\ln(\csc(theta2)+\cot(theta2))$$
$$\frac{(\cos(\theta 2)+\cos(\theta 1))^2}{-\cos(\theta 1)^3+3\cos(\theta 1)+3\cos(\theta 2)-\cos(\theta 2)^3}+\ln(\csc(\theta 1)+\cot(\theta 1))+\ln(\csc(\theta 2)) \tag{33}$$
$$+\cot(\theta 2))$$

Changing sign (so the expression is for

$\frac{2\cdot \mathrm{Pi}\cdot \mathrm{gamma}}{k}$) and applying an identity:

$$applyrule\left(-\ln(\csc(x::anything)+\cot(x::anything))=\ln\left(\tan\left(\frac{x}{2}\right)\right),-(33)\right)$$

$$-\frac{(\cos(\theta 2)+\cos(\theta 1))^2}{-\cos(\theta 1)^3+3\cos(\theta 1)+3\cos(\theta 2)-\cos(\theta 2)^3}+\ln\left(\tan\left(\frac{\theta 1}{2}\right)\right)+\ln\left(\tan\left(\frac{\theta 2}{2}\right)\right) \quad (34)$$

For the first term, move the minus to the denominator and factor:

$$factor\left(-\left(3\cos(\theta 1)-\cos(\theta 1)^3+3\cos(\theta 2)-\cos(\theta 2)^3\right)\right)$$

$$(\cos(\theta 2)+\cos(\theta 1))\left(\cos(\theta 1)^2-\cos(\theta 2)\cos(\theta 1)+\cos(\theta 2)^2-3\right) \quad (35)$$

After canceling out the common factor, this is exactly the same expression as before.

For the **case II**, Kusumaatmaja & Lipowsky gave the expression for $-\frac{2\cdot \mathrm{Pi}\cdot \mathrm{gamma}}{k}$ as

$$\frac{(\cos(theta1)+\cos(theta2))\cdot\cos(theta1)}{(3\cdot\cos(theta1)-\cos(theta1)^3+3\cdot\cos(theta2)-\cos(theta2)^3)}+\frac{\cos(theta2)}{\sin(theta2)^2}+\ln(\csc(theta1)+\cot(theta1))+\ln(\csc(theta2)+\cot(theta2))-\left(\left(\frac{2}{\sin(theta2)}-\frac{2\cdot\sin(theta2)^3\cdot\cos(theta1)}{(3\cdot\cos(theta1)-\cos(theta1)^3+3\cdot\cos(theta2)-\cos(theta2)^3)}\right)\cdot\left(2\cdot\cot(theta2)+\frac{2\cdot\cos(theta1)}{\sin(theta2)}+\frac{\sin(theta1)^2\cdot\cos(theta1)}{\sin(theta2)}-\sin(theta2)\cdot\cos(theta1)\right)\right)\Big/(2+2\cdot\cos(theta2)^2+4\cdot\cos(theta2)\cdot\cos(theta1)+2\cdot\sin(theta1)^2\cdot\cos(theta1)\cdot\cos(theta2))$$

$$\frac{(\cos(\theta 2)+\cos(\theta 1))\cos(\theta 1)}{-\cos(\theta 1)^3+3\cos(\theta 1)+3\cos(\theta 2)-\cos(\theta 2)^3}+\frac{\cos(\theta 2)}{\sin(\theta 2)^2}+\ln(\csc(\theta 1)+\cot(\theta 1))+\ln(\csc(\theta 2)+\cot(\theta 2))-\left(\left(\frac{2}{\sin(\theta 2)}-\frac{2\sin(\theta 2)^3\cos(\theta 1)}{-\cos(\theta 1)^3+3\cos(\theta 1)+3\cos(\theta 2)-\cos(\theta 2)^3}\right)\left(2\cot(\theta 2)+\frac{2\cos(\theta 1)}{\sin(\theta 2)}+\frac{\sin(\theta 1)^2\cos(\theta 1)}{\sin(\theta 2)}-\sin(\theta 2)\cos(\theta 1)\right)\right)\Big/(2+2\cos(\theta 2)^2+4\cos(\theta 2)\cos(\theta 1)+2\cos(\theta 1)\cos(\theta 2)\sin(\theta 1)^2) \quad (36)$$

Clearly, the logarithm are the same as before so focus on the rational part

$$\frac{(\cos(theta1)+\cos(theta2))\cdot\cos(theta1)}{(3\cdot\cos(theta1)-\cos(theta1)^3+3\cdot\cos(theta2)-\cos(theta2)^3)}+\frac{\cos(theta2)}{\sin(theta2)^2}-\left(\left(\frac{2}{\sin(theta2)}-\frac{2\cdot\sin(theta2)^3\cdot\cos(theta1)}{(3\cdot\cos(theta1)-\cos(theta1)^3+3\cdot\cos(theta2)-\cos(theta2)^3)}\right)\cdot\left(2\cdot\cot(theta2)+\frac{2\cdot\cos(theta1)}{\sin(theta2)}+\frac{\sin(theta1)^2\cdot\cos(theta1)}{\sin(theta2)}-\sin(theta2)\cdot\cos(theta1)\right)\right)\Big/(2+2\cdot\cos(theta2)^2+4\cdot\cos(theta2)\cdot\cos(theta1)+2\cdot\sin(theta1)^2\cdot\cos(theta1)\cdot\cos(theta2))$$

$$\frac{(\cos(\theta 2)+\cos(\theta 1))\cos(\theta 1)}{-\cos(\theta 1)^3+3\cos(\theta 1)+3\cos(\theta 2)-\cos(\theta 2)^3}+\frac{\cos(\theta 2)}{\sin(\theta 2)^2}-\left(\left(\frac{2}{\sin(\theta 2)}-\frac{2\sin(\theta 2)^3\cos(\theta 1)}{-\cos(\theta 1)^3+3\cos(\theta 1)+3\cos(\theta 2)-\cos(\theta 2)^3}\right)\left(2\cot(\theta 2)+\frac{2\cos(\theta 1)}{\sin(\theta 2)}+\frac{\sin(\theta 1)^2\cos(\theta 1)}{\sin(\theta 2)}-\sin(\theta 2)\cos(\theta 1)\right)\right)\Big/\left(2+2\cos(\theta 2)^2+4\cos(\theta 2)\cos(\theta 1)+2\cos(\theta 1)\cos(\theta 2)\sin(\theta 1)^2\right) \quad \textbf{(37)}$$

Changing sign (so the expression is for $\frac{2\cdot \mathrm{Pi}\cdot \mathrm{gamma}}{k}$) and simplifying

simplify(-**(37)**)

$$-\frac{(\cos(\theta 2)+\cos(\theta 1))(\cos(\theta 2)^2-1)(\cos(\theta 1)^2-1)}{\sin(\theta 2)^2(-1-\cos(\theta 2)^2+(\cos(\theta 1)^3-3\cos(\theta 1))\cos(\theta 2))} \quad \textbf{(38)}$$

The $\sin(theta2)^2$ denominator cancels out, leaving a minus sign:

simplify(*applyrule*($\cos(_x::anything)^2-1=-\sin(_x)^2$, **(38)**))

$$-\frac{(\cos(\theta 2)+\cos(\theta 1))\sin(\theta 1)^2}{-1-\cos(\theta 2)^2+(\cos(\theta 1)^3-3\cos(\theta 1))\cos(\theta 2)} \quad \textbf{(39)}$$

From here, the equation can be brought to the same form as before with little effort.

Finally, for the **case III**, Kusumaatmaja & Lipowsky gave the expression for $-\frac{2\cdot \mathrm{Pi}\cdot \mathrm{gamma}}{k}$ by first defining function f

$$f:=(a,b,Ra,Rb)\to\left(2\cdot\cot(a)+\frac{2\cdot\cos(b)}{\sin(a)}+\frac{\sin(b)^2\cdot\cos(b)}{\sin(a)}-\sin(a)\cdot\cos(b)+\frac{2\cdot\sin(b)^3\cdot\sin(a)\cdot(Rb\cdot\csc(a)-Ra\cdot\csc(b))}{2\cdot Ra\cdot\cos(b)}\right)\Big/\left(2+2\cdot\cos(a)^2+4\cdot\cos(a)\cdot\cos(b)+2\cdot\sin(b)^2\cdot\cos(b)\cdot\cos(a)+\frac{2\cdot\sin(b)^3\cdot\sin(a)\cdot(Rb\cdot\cos(a))}{Ra\cdot\cos(b)}\right)$$

$$f:=(a,b,Ra,Rb)\mapsto(2\cot(a)+2\cos(b)\sin(a)^{-1}+\sin(b)^2\cos(b)\sin(a)^{-1}+(-\sin(a)\cos(b))+2\sin(b)^3\sin(a)(Rb\csc(a)+(-Ra\csc(b)))(2Ra\cos(b))^{-1})(2+2\cos(a)^2+4\cos(a)\cos(b)+2\sin(b)^2\cos(b)\cos(a)+2\sin(b)^3\sin(a)(Rb\cos(a))(Ra\cos(b))^{-1})^{-1} \quad \textbf{(40)}$$

And then the $-\frac{2\cdot \mathrm{Pi}\cdot \mathrm{gamma}}{k}$ as a

$$-\frac{2}{\sin(theta1)}f(theta1,theta2,R1,R2)-\frac{2}{\sin(theta2)}f(theta2,theta1,R2,R1)+\frac{\cos(theta1)}{\sin(theta1)^2}+\frac{\cos(theta2)}{\sin(theta2)^2}+\ln(\csc(theta1)+\cot(theta1))+\ln(\csc(theta2)+\cot(theta2))$$

$$-\left(2\left(2\cot(\theta 1)+\frac{2\cos(\theta 2)}{\sin(\theta 1)}+\frac{\sin(\theta 2)^2\cos(\theta 2)}{\sin(\theta 1)}-\sin(\theta 1)\cos(\theta 2)\right)\right. \quad \textbf{(41)}$$

$$+\frac{\sin(\theta 2)^3\sin(\theta 1)\,(R2\csc(\theta 1)-R1\csc(\theta 2))}{R1\cos(\theta 2)}\Bigg)\Bigg)\Big/\Bigg(\sin(\theta 1)\Bigg(2+2\cos(\theta 1)^2$$

$$+4\cos(\theta 2)\cos(\theta 1)+2\sin(\theta 2)^2\cos(\theta 1)\cos(\theta 2)$$

$$+\frac{2\sin(\theta 2)^3\sin(\theta 1)\,R2\cos(\theta 1)}{R1\cos(\theta 2)}\Bigg)\Bigg)-\Bigg(2\Bigg(2\cot(\theta 2)+\frac{2\cos(\theta 1)}{\sin(\theta 2)}$$

$$+\frac{\sin(\theta 1)^2\cos(\theta 1)}{\sin(\theta 2)}-\sin(\theta 2)\cos(\theta 1)$$

$$+\frac{\sin(\theta 1)^3\sin(\theta 2)\,(R1\csc(\theta 2)-R2\csc(\theta 1))}{R2\cos(\theta 1)}\Bigg)\Bigg)\Big/\Bigg(\sin(\theta 2)\Bigg(2+2\cos(\theta 2)^2$$

$$+4\cos(\theta 2)\cos(\theta 1)+2\cos(\theta 1)\cos(\theta 2)\sin(\theta 1)^2$$

$$+\frac{2\sin(\theta 1)^3\sin(\theta 2)\,R1\cos(\theta 2)}{R2\cos(\theta 1)}\Bigg)\Bigg)+\frac{\cos(\theta 1)}{\sin(\theta 1)^2}+\frac{\cos(\theta 2)}{\sin(\theta 2)^2}+\ln(\csc(\theta 1)$$

$$+\cot(\theta 1))+\ln(\csc(\theta 2)+\cot(\theta 2))$$

Kusutmaaja & Lipowsky noted that $\frac{R1}{R2}=\frac{\sin(theta1)}{\sin(theta2)}$, put that into **(41)**

$$simplify\left(subs\left(R1=R2\cdot\frac{\sin(theta1)}{\sin(theta2)},\ \textbf{(41)}\right)\right)$$

$$\frac{1}{(\cos(\theta 2)\cos(\theta 1)+1)\sin(\theta 2)^2\sin(\theta 1)^2}\Bigg(\Bigg(\ln\left(\frac{1+\cos(\theta 1)}{\sin(\theta 1)}\right)\cos(\theta 1)\cos(\theta 2)+\ln\left(\frac{\cos(\theta 2)+1}{\sin(\theta 2)}\right)\cos(\theta 1)\cos(\theta 2)+\ln\left(\frac{1+\cos(\theta 1)}{\sin(\theta 1)}\right)+\ln\left(\frac{\cos(\theta 2)+1}{\sin(\theta 2)}\right)$$

$$-\cos(\theta 1)-\cos(\theta 2)\Bigg)(\cos(\theta 2)^2-1)(\cos(\theta 1)^2-1)\Bigg)\qquad\textbf{(42)}$$

$simplify(applyrule(\sin(a::anything)=\mathrm{sqrt}(1-\cos(a)^2),\ \textbf{(42)}))$ assuming $theta1\geq 0$, $theta1\leq \mathrm{Pi}$, $theta2\geq 0$, $theta2\leq \mathrm{Pi}$

$$\frac{1}{\cos(\theta 2)\cos(\theta 1)+1}((\ln(1+\cos(\theta 1))-\ln(\sin(\theta 1)))\cos(\theta 1)\cos(\theta 2)+(\ln(\cos(\theta 2)+1)-\ln(\sin(\theta 2)))\cos(\theta 1)\cos(\theta 2)+\ln(1+\cos(\theta 1))-\ln(\sin(\theta 1))+\ln(\cos(\theta 2)+1)-\ln(\sin(\theta 2))-\cos(\theta 1)-\cos(\theta 2))\qquad\textbf{(43)}$$

Change sign so the expression is for $\frac{2\cdot \mathrm{Pi}\cdot gamma}{k}$

$combine(collect(-\textbf{(43)},\ln),\ln)$

$$-\ln(1+\cos(\theta 1))-\ln(\cos(\theta 2)+1)-\frac{(-\cos(\theta 2)\cos(\theta 1)-1)\ln(\sin(\theta 1))}{\cos(\theta 2)\cos(\theta 1)+1}\qquad\textbf{(44)}$$

$$+\ln(\sin(\theta 2)) - \frac{-\cos(\theta 2) - \cos(\theta 1)}{\cos(\theta 2)\cos(\theta 1) + 1}$$

which is what we were after.

Next, we will derive the quadratic spring constants. Let's first give a short hand for the Jacobian determinant:

```
with(Student) :
with(MultivariateCalculus) :
J := (x, y) → Jacobian(x, y, output = determinant)
```

$$J := (x, y) \mapsto Jacobian(x, y, output = determinant) \quad \textbf{(45)}$$

Then we define a operator for taking the constrained derivative (the signs alternate because of the rules of swapping a column in determinant):

```
diff_h := (x, c) → 1/(J([h, op(c)], [B, R, alpha1, alpha2])) (simplify((diff(x, B)·J(c, [R, alpha1,
    alpha2]) - diff(x, R)·J(c, [B, alpha1, alpha2]) + diff(x, alpha1)·J(c, [B, R, alpha2]) - diff(x,
    alpha2)·J(c, [B, R, alpha1]))))
```

$$diff_h := (x, c) \to VectorCalculus:-`*`(simplify(VectorCalculus:-`+`(VectorCalculus:-`+`(VectorCalculus:-`+`(VectorCalculus:-`*`(Student:-MultivariateCalculus:-diff(x, B), J(c, [R, \alpha 1, \alpha 2])), VectorCalculus:-`-`(VectorCalculus:-`*`(Student:-MultivariateCalculus:-diff(x, R), J(c, [B, \alpha 1, \alpha 2]))))), VectorCalculus:-`*`(Student:-MultivariateCalculus:-diff(x, \alpha 1), J(c, [B, R, \alpha 2])))), VectorCalculus:-`-`(VectorCalculus:-`*`(Student:-MultivariateCalculus:-diff(x, \alpha 2), J(c, [B, R, \alpha 1]))))), J([h, op(c)], [B, R, \alpha 1, \alpha 2])^{VectorCalculus:-`-`(1)}) \quad \textbf{(46)}$$

In the case III, the quadratic spring constant (equation (21) in the paper) is:

```
k2_caseIII := -1/2 · simplify(subs(subs(B = 1, diff_h(diff_h((13), [V, r1,
    r2]), [V, r1, r2]))))
```

$$k2_caseIII := -\left(3\left((\cos(\theta 2)\cos(\theta 1) + 1)^3 \ln(1 - \cos(\theta 2)) - (\cos(\theta 2)\cos(\theta 1) + 1)^3 \ln(1 + \cos(\theta 1)) + (\cos(\theta 2)\cos(\theta 1) + 1)^3 \ln(\sin(\theta 1)) - (\cos(\theta 2)\cos(\theta 1) + 1)^3 \ln(\sin(\theta 2)) + \left(\left(\cos(\theta 2)^2 + \frac{1}{3}\right)\cos(\theta 1)^2 + \frac{8\cos(\theta 2)\cos(\theta 1)}{3} + \frac{\cos(\theta 2)^2}{3} + 1\right)(\cos(\theta 2) + \cos(\theta 1))\right)\gamma\pi\right) \Big/ \left(R\left(\ln(\sin(\theta 1))\cos(\theta 1)\cos(\theta 2) - \ln(\sin(\theta 2))\cos(\theta 1)\cos(\theta 2) - \ln(1 + \cos(\theta 1))\cos(\theta 1)\cos(\theta 2) + \ln(1 - \cos(\theta 2))\cos(\theta 1)\cos(\theta 2) + \ln(\sin(\theta 1)) - \ln(\sin(\theta 2)) - \ln(1 + \cos(\theta 1)) + \ln(1 - \cos(\theta 2)) + \cos(\theta 1) + \cos(\theta 2)\right)^3\right)$$

Next we replace some of the subexpressions to simplify the expression:

$$algsubs(-\ln(1-\cos(\theta 2))+\ln(1+\cos(\theta 1))-\ln(\sin(\theta 1))+\ln(\sin(\theta 2))=C,\ \textbf{(47)})$$

$$-\left(3\,\gamma\pi\left(-C\cos(\theta 1)^3\cos(\theta 2)^3-3\,C\cos(\theta 1)^2\cos(\theta 2)^2-3\,C\cos(\theta 1)\cos(\theta 2)-C+\cos(\theta 2)^3\cos(\theta 1)^2+\cos(\theta 2)^2\cos(\theta 1)^3+3\cos(\theta 1)^2\cos(\theta 2)+\frac{\cos(\theta 1)^3}{3}+3\cos(\theta 2)^2\cos(\theta 1)+\frac{\cos(\theta 2)^3}{3}+\cos(\theta 2)+\cos(\theta 1)\right)\right)\Big/\left(R\,(-C\cos(\theta 1)\cos(\theta 2)-C+\cos(\theta 1)+\cos(\theta 2))^3\right) \quad \textbf{(48)}$$

$$Yeq := \cos(theta1)\cdot\cos(theta2)+1=Y;$$

$$Yeq := \cos(\theta 2)\cos(\theta 1)+1=Y \quad \textbf{(49)}$$

$$Xeq := \cos(theta1)+\cos(theta2)=X;$$

$$Xeq := \cos(\theta 2)+\cos(\theta 1)=X \quad \textbf{(50)}$$

$$algsubs(Yeq^3, collect(\textbf{(48)}, C))$$

$$-\left(3\,\gamma\pi\left(-C\,Y^3+\cos(\theta 2)^3\cos(\theta 1)^2+\cos(\theta 2)^2\cos(\theta 1)^3+3\cos(\theta 1)^2\cos(\theta 2)+\frac{\cos(\theta 1)^3}{3}+3\cos(\theta 2)^2\cos(\theta 1)+\frac{\cos(\theta 2)^3}{3}+\cos(\theta 2)+\cos(\theta 1)\right)\right)\Big/\left(R\,((-\cos(\theta 2)\cos(\theta 1)-1)\,C+\cos(\theta 2)+\cos(\theta 1))^3\right) \quad \textbf{(51)}$$

$$algsubs(3\cdot Yeq^2\cdot Xeq+Xeq^3,\ \textbf{(51)})$$

$$-\frac{3\,\gamma\pi\left(-C\,Y^3+\frac{1}{3}X^3+Y^2X\right)}{R\,((-\cos(\theta 2)\cos(\theta 1)-1)\,C+\cos(\theta 2)+\cos(\theta 1))^3} \quad \textbf{(52)}$$

$$k2_caseIII_simplified := factor(algsubs(Xeq, algsubs(Yeq, \textbf{(52)})))$$

$$k2_caseIII_simplified := -\frac{\gamma\pi\,(3\,C\,Y^3-X^3-3\,Y^2X)}{R\,(C\,Y-X)^3} \quad \textbf{(53)}$$

which is finally equation 21 in the paper.

In the case II, the quadratic spring constant is:

$$k2_caseII := -\frac{1}{2}\cdot simplify(subs(\textbf{(15)}, \textbf{(16)}, simplify(value(subs(B=1, diff_h(diff_h(\textbf{(13)}, [V, slope1, r2]), [V, slope1, r2])))))))$$

$$k2_caseII := \left(3\left(\left(-1-\cos(\theta 2)\right)^2+\left(\cos(\theta 1)^3-3\cos(\theta 1)\right)\cos(\theta 2)\right)^3(1+\cos(\theta 1))\,(\cos(\theta 1)-1)\ln(1-\cos(\theta 2))-\left(-1-\cos(\theta 2)\right)^2+\left(\cos(\theta 1)^3\right. \quad \textbf{(54)}$$

$$-3\cos(\theta 1))\cos(\theta 2))^3(1+\cos(\theta 1))(\cos(\theta 1)-1)\ln(1+\cos(\theta 1))+(-1$$

$$-\cos(\theta 2)^2+(\cos(\theta 1)^3-3\cos(\theta 1))\cos(\theta 2))^3(1+\cos(\theta 1))(\cos(\theta 1)$$

$$-1)\ln(\sin(\theta 1))-(-1-\cos(\theta 2)^2+(\cos(\theta 1)^3-3\cos(\theta 1))\cos(\theta 2))^3(1$$

$$+\cos(\theta 1))(\cos(\theta 1)-1)\ln(\sin(\theta 2))+\left(-\cos(\theta 1)\cos(\theta 2)^5+(3\cos(\theta 1)^4\right.$$

$$-9\cos(\theta 1)^2+1)\cos(\theta 2)^4+(-3\cos(\theta 1)^7+19\cos(\theta 1)^5-31\cos(\theta 1)^3$$

$$+5\cos(\theta 1))\cos(\theta 2)^3+\left(\frac{10}{3}+\frac{95\cos(\theta 1)^6}{3}-33\cos(\theta 1)^4-\frac{10\cos(\theta 1)^2}{3}\right.$$

$$\left.-\frac{29\cos(\theta 1)^8}{3}+\cos(\theta 1)^{10}\right)\cos(\theta 2)^2$$

$$-\frac{\cos(\theta 1)(\cos(\theta 1)^8+5\cos(\theta 1)^6-48\cos(\theta 1)^4+83\cos(\theta 1)^2-26)\cos(\theta 2)}{3}+1$$

$$\left.-\frac{7\cos(\theta 1)^8}{3}+6\cos(\theta 1)^6-\frac{2\cos(\theta 1)^2}{3}+\frac{\cos(\theta 1)^{10}}{3}-\frac{16\cos(\theta 1)^4}{3}\right)(\cos(\theta 2)$$

$$\left.+\cos(\theta 1))\right)\gamma\pi\Big)\Big/\Big(((-1-\cos(\theta 2)^2+(\cos(\theta 1)^3-3\cos(\theta 1))\cos(\theta 2))\ln(1$$

$$-\cos(\theta 2))+(1+\cos(\theta 2)^2+(-\cos(\theta 1)^3+3\cos(\theta 1))\cos(\theta 2))\ln(1+\cos(\theta 1))$$

$$+(-1-\cos(\theta 2)^2+(\cos(\theta 1)^3-3\cos(\theta 1))\cos(\theta 2))\ln(\sin(\theta 1))+(1+\cos(\theta 2)^2$$

$$+(-\cos(\theta 1)^3+3\cos(\theta 1))\cos(\theta 2))\ln(\sin(\theta 2))+\cos(\theta 1)^3+\cos(\theta 1)^2\cos(\theta 2)$$

$$-\cos(\theta 1)-\cos(\theta 2))^3R\sin(\theta 1)^2\Big)$$

We can simplify the expression at least slightly:

collect(*simplify*(*algsubs*($-\ln(1-\cos(\theta 2))+\ln(1+\cos(\theta 1))-\ln(\sin(\theta 1))+\ln(\sin(\theta 2))=C$, **(54)**)), C)

$$\Big(3\Big((\cos(\theta 1)^{11}\cos(\theta 2)^3-10\cos(\theta 1)^9\cos(\theta 2)^3+(-3\cos(\theta 2)^4-3\cos(\theta 2)^2)\cos(\theta 1)^8 \qquad \textbf{(55)}$$

$$+36\cos(\theta 1)^7\cos(\theta 2)^3+(21\cos(\theta 2)^4+21\cos(\theta 2)^2)\cos(\theta 1)^6+(3\cos(\theta 2)^5$$

$$-48\cos(\theta 2)^3+3\cos(\theta 2))\cos(\theta 1)^5+(-45\cos(\theta 2)^4-45\cos(\theta 2)^2)\cos(\theta 1)^4+($$

$$-12\cos(\theta 2)^5+3\cos(\theta 2)^3-12\cos(\theta 2))\cos(\theta 1)^3+(-\cos(\theta 2)^6+24\cos(\theta 2)^4$$

$$+24\cos(\theta 2)^2-1)\cos(\theta 1)^2+(9\cos(\theta 2)^5+18\cos(\theta 2)^3+9\cos(\theta 2))\cos(\theta 1)+1$$

$$+\cos(\theta 2)^6+3\cos(\theta 2)^4+3\cos(\theta 2)^2)C+\left(-\frac{1}{3}-\cos(\theta 2)^2\right)\cos(\theta 1)^{11}$$

$$-\cos(\theta 1)^{10}\cos(\theta 2)^3+\left(\frac{7}{3}+10\cos(\theta 2)^2\right)\cos(\theta 1)^9+\left(\frac{38\cos(\theta 2)^3}{3}\right.$$

$$\left.+4\cos(\theta 2)\right)\cos(\theta 1)^8+(3\cos(\theta 2)^4-30\cos(\theta 2)^2-6)\cos(\theta 1)^7+\left(-22\cos(\theta 2)\right.$$

$$\left.-\frac{152\cos(\theta 2)^3}{3}\right)\cos(\theta 1)^6+\left(\frac{16}{3}-22\cos(\theta 2)^4+17\cos(\theta 2)^2\right)\cos(\theta 1)^5+($$

$$-3\cos(\theta 2)^5+64\cos(\theta 2)^3+33\cos(\theta 2))\cos(\theta 1)^4+\left(\frac{2}{3}+40\cos(\theta 2)^4\right.$$

$$\left.+31\cos(\theta 2)^2\right)\cos(\theta 1)^3+\left(-8\cos(\theta 2)+10\cos(\theta 2)^5-\frac{5\cos(\theta 2)^3}{3}\right)\cos(\theta 1)^2$$

$$+(\cos(\theta 2)^6-6\cos(\theta 2)^4-12\cos(\theta 2)^2-1)\cos(\theta 1)-\cos(\theta 2)-\frac{10\cos(\theta 2)^3}{3}$$

$$\left.-\cos(\theta 2)^5\right)\gamma\pi\Big)\Big/\Big(\sin(\theta 1)^2((\cos(\theta 1)^3\cos(\theta 2)-3\cos(\theta 2)\cos(\theta 1)$$

$$-\cos(\theta 2)^2-1)C-\cos(\theta 1)^3-\cos(\theta 1)^2\cos(\theta 2)+\cos(\theta 1)+\cos(\theta 2))^3R\Big)$$

Finally, for the case I, the quadratic spring constant is:

$k2_caseI := -\frac{1}{2}\cdot simplify(\ subs(\mathbf{(15)}, \mathbf{(16)}, simplify(value(subs(B=1, diff_h(diff_h(\mathbf{(13)}, [V, slope1, slope2]), [V, slope1, slope2]))))))$

$$k2_caseI := -\Big(3\gamma\pi\Big((\cos(\theta 2)-1)(\cos(\theta 2)+1)(\cos(\theta 1)-1)(1 \tag{56}$$

$$+\cos(\theta 1))(\cos(\theta 1)^2-\cos(\theta 2)\cos(\theta 1)+\cos(\theta 2)^2-3)^2\ln(1-\cos(\theta 2))$$

$$-(\cos(\theta 2)-1)(\cos(\theta 2)+1)(\cos(\theta 1)-1)(1+\cos(\theta 1))(\cos(\theta 1)^2$$

$$-\cos(\theta 2)\cos(\theta 1)+\cos(\theta 2)^2-3)^2\ln(1+\cos(\theta 1))+(\cos(\theta 2)-1)(\cos(\theta 2)$$

$$+1)(\cos(\theta 1)-1)(1+\cos(\theta 1))(\cos(\theta 1)^2-\cos(\theta 2)\cos(\theta 1)+\cos(\theta 2)^2$$

$$-3)^2\ln(\sin(\theta 1))-(\cos(\theta 2)-1)(\cos(\theta 2)+1)(\cos(\theta 1)-1)(1$$

$$+\cos(\theta 1))(\cos(\theta 1)^2-\cos(\theta 2)\cos(\theta 1)+\cos(\theta 2)^2-3)^2\ln(\sin(\theta 2))+(\cos(\theta 2)$$

$$+\cos(\theta 1))\left(\cos(\theta 1)^5\cos(\theta 2)+\left(-2\cos(\theta 2)^2-1\right)\cos(\theta 1)^4+\left(3\cos(\theta 2)^3\right.\right.$$
$$\left.-4\cos(\theta 2)\right)\cos(\theta 1)^3+\left(-2\cos(\theta 2)^4+\frac{7\cos(\theta 2)^2}{3}+\frac{20}{3}\right)\cos(\theta 1)^2+\left(\cos(\theta 2)^5\right.$$
$$\left.\left.-4\cos(\theta 2)^3+3\cos(\theta 2)\right)\cos(\theta 1)-\cos(\theta 2)^4+\frac{20\cos(\theta 2)^2}{3}-\frac{29}{3}\right)\left(\cos(\theta 1)^2\right.$$
$$\left.-\cos(\theta 2)\cos(\theta 1)+\cos(\theta 2)^2-3\right)\Big/\left(\sin(\theta 2)^2 R\left(\left(\cos(\theta 1)^2\right.\right.\right.$$
$$-\cos(\theta 2)\cos(\theta 1)+\cos(\theta 2)^2-3)\ln(1-\cos(\theta 2))+\left(-\cos(\theta 1)^2+\cos(\theta 2)\cos(\theta 1)\right.$$
$$\left.-\cos(\theta 2)^2+3\right)\ln(1+\cos(\theta 1))+\left(\cos(\theta 1)^2-\cos(\theta 2)\cos(\theta 1)+\cos(\theta 2)^2\right.$$
$$-3)\ln(\sin(\theta 1))+\left(-\cos(\theta 1)^2+\cos(\theta 2)\cos(\theta 1)-\cos(\theta 2)^2+3\right)\ln(\sin(\theta 2))$$
$$\left.\left.+\cos(\theta 2)+\cos(\theta 1)\right)^3\sin(\theta 1)^2\right)$$

$collect(simplify(algsubs(-\ln(1-\cos(\theta 2))+\ln(1+\cos(\theta 1))-\ln(\sin(\theta 1))+\ln(\sin(\theta 2))=C,$ **(56)**$),C)$

$$-\left(3\left(\cos(\theta 1)^2-\cos(\theta 2)\cos(\theta 1)+\cos(\theta 2)^2-3\right)\gamma\pi\left(\left(\left(\cos(\theta 2)^2-1\right)\cos(\theta 1)^6+\right.\right.\right. \quad \textbf{(57)}$$
$$\left(-2\cos(\theta 2)^3+2\cos(\theta 2)\right)\cos(\theta 1)^5+\left(3\cos(\theta 2)^4-10\cos(\theta 2)^2+7\right)\cos(\theta 1)^4+\left(\right.$$
$$\left.-2\cos(\theta 2)^5+10\cos(\theta 2)^3-8\cos(\theta 2)\right)\cos(\theta 1)^3+\left(\cos(\theta 2)^6-10\cos(\theta 2)^4\right.$$
$$\left.+24\cos(\theta 2)^2-15\right)\cos(\theta 1)^2+\left(2\cos(\theta 2)^5-8\cos(\theta 2)^3+6\cos(\theta 2)\right)\cos(\theta 1)+9$$
$$-15\cos(\theta 2)^2-\cos(\theta 2)^6+7\cos(\theta 2)^4)C-\cos(\theta 2)\cos(\theta 1)^6+\left(\cos(\theta 2)^2\right.$$
$$+1)\cos(\theta 1)^5+\left(-\cos(\theta 2)^3+5\cos(\theta 2)\right)\cos(\theta 1)^4+\left(-\cos(\theta 2)^4+\frac{5\cos(\theta 2)^2}{3}\right.$$
$$\left.-\frac{20}{3}\right)\cos(\theta 1)^3+\left(\frac{5\cos(\theta 2)^3}{3}-\frac{29\cos(\theta 2)}{3}+\cos(\theta 2)^5\right)\cos(\theta 1)^2+\left(\frac{29}{3}\right.$$
$$\left.-\cos(\theta 2)^6+5\cos(\theta 2)^4-\frac{29\cos(\theta 2)^2}{3}\right)\cos(\theta 1)+\frac{29\cos(\theta 2)}{3}-\frac{20\cos(\theta 2)^3}{3}$$
$$\left.+\cos(\theta 2)^5\right)\Big)\Big/\left(\sin(\theta 1)^2 R\sin(\theta 2)^2\left(\left(\cos(\theta 1)^2-\cos(\theta 2)\cos(\theta 1)+\cos(\theta 2)^2\right.\right.\right.$$
$$\left.\left.-3)C-\cos(\theta 1)-\cos(\theta 2)\right)^3\right)$$