Supporting information:

Semi-analytical model for design and analysis of grating-assisted radiation emission of quantum emitters in hyperbolic metamaterials

Achiles F. da Mota[1,2], Augusto Martins[1], Heidi Ottevaere[2], Wendy Meulebroeck[2], Emiliano R. Martins[1], John Weiner[4], Fernando L. Teixeira[3], and Ben-Hur V. Borges[1,*].

[1] Department of Electrical and Computer Engineering, University of São Paulo, CEP 13566-590, São Carlos, SP, Brazil.
[2] Brussels Photonics Team, Department of Applied Physics and Photonics, Vrije Universiteit Brussel, 1050 Brussels, Belgium.
[3] ElectroScience Laboratory, The Ohio State University, Columbus, Ohio 43210, USA.
[4] Physics Institute of São Carlos, University of São Paulo, CEP 13566-590, São Carlos, SP, Brazil.
[*] Corresponding author: benhur@sc.usp.br

The supporting information contains 14 pages and 4 figures.

In this Supporting Material we show how the equations presented in the main text are derived. We start with the Purcell factor and the dipole radiated power for a three-dimensional (3D) grating, both discussed in section 1. Since 3D simulations are computationally costly, particularly if used in the optimization of grating parameters, we present in section 2 a simplified two-dimensional (2D) formalism for a dipole interacting with a 2D grating. This 2D method allows us not only to obtain the reflection and transmission coefficients for all diffraction orders but also to more efficiently optimize the grating parameters for controlling the dipole's radiation pattern.

1. 3D grating

The geometry adopted in the 3D method is illustrated in Figure S1, where the quantum emitters (QE) is treated as a dipole embedded in medium 1 (with permittivity ε_1). The QE is centered at the origin at a distance q from the hyperbolic metamaterial's (HMM's) bottom surface and z_{gr} from the grating upper surface. The metallic grating with periodicities Λ_x and Λ_y in x and y directions, respectively, is on top of the HMM and is covered with medium 2 (with permittivity ε_2). In the next section, we calculate the transmitted and reflected electric fields which are required to obtain the Purcell factor and the dipole's radiated power.

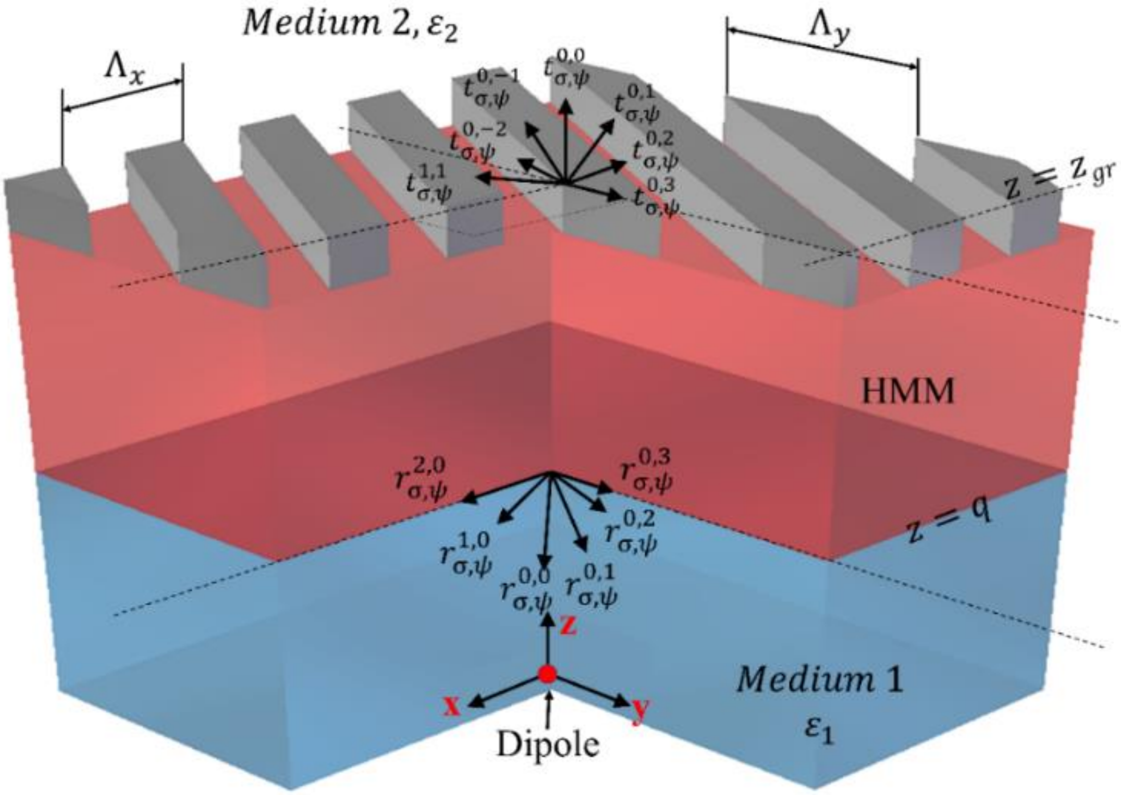

Figure S1. Proposed 3D geometry in which the QEs are modelled as a dipole embedded in medium 1 (with permittivity ε_1), centered at the origin and at a distance q from the HMM bottom surface. The grating with periodicities Λ_x and Λ_y in x and y directions, respectively, on top of the HMM is covered with medium 2 (with permittivity ε_2).

1.1. Calculation of the transmitted and reflected electric fields

Assuming temporal dependence $e^{-j\omega t}$, the current generated by a dipole source centered at the origin and embedded in medium 1 ($\mathbf{J}$) is calculated as follows [1],

$$\mathbf{J}(\mathbf{r}) = -\mathrm{j}\omega\delta^{(3)}(\mathbf{r})\mathbf{p}, \tag{1}$$

where $\mathbf{r} = x\mathbf{x} + y\mathbf{y} + z\mathbf{z}$, $\delta^{(3)}(\mathbf{r})$ is the 3D Dirac delta function, $\mathbf{p}$ is the dipole moment and ω is the angular frequency. The electric field $\mathbf{E_i}$ at $\mathbf{r}$ generated by $\mathbf{J}$ is then written as

$$\mathbf{E_i}(\mathbf{r}) = \omega^2 \mu_1 \left(\mathbf{I} + \frac{1}{k_1^2} \nabla\nabla \right) \frac{\mathrm{e}^{\mathrm{j}k_1|\mathbf{r}|}}{4\pi|\mathbf{r}|} \mathbf{p}. \tag{2}$$

where μ_1 is the permeability of medium 1, $\mathbf{I}$ is the identity tensor and $k_1 = |\mathbf{k_1}| = |k_x\mathbf{x} + k_y\mathbf{y} + k_{z1}\mathbf{z}|$ is the wavenumber vector magnitude in medium 1. Using the Weyl identity

$$\frac{e^{\mathrm{j}k_1|\mathbf{r}|}}{4\pi|\mathbf{r}|} = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int \frac{\mathrm{j}}{4\pi k_{z1}} e^{\mathrm{j}\left(k_x x + k_y y \pm k_{z1} z\right)} \mathrm{d}k_x \mathrm{d}k_y, \tag{3}$$

in (2), we obtain,

$$\mathbf{E_i}(\mathbf{r}) = \omega^2 \mu_1 \left(\mathbf{I} + \frac{1}{k_1^2} \nabla\nabla \right) \frac{1}{2\pi} \int_{-\infty}^{\infty}\int \frac{\mathrm{j}}{4\pi k_{z1}} \mathrm{e}^{\mathrm{j}\left(k_x x + k_y y \pm k_{z1} z\right)} \mathrm{d}k_x \mathrm{d}k_y \mathbf{p}. \tag{4}$$

The spatial differentials are more easily considered if $\mathbf{E_i}$ is Fourier transformed (FT). The FT and the inverse FT are defined as

$$\mathbf{E_i^{FT}}\left(\mathbf{k}_{||}, z\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int \mathbf{E_i}(\mathbf{r}) \mathrm{e}^{-\mathrm{j}\left(k_x x + k_y y\right)} \mathrm{d}x\mathrm{d}y, \tag{5a}$$

$$\mathbf{E_i}(\mathbf{r}) = \frac{1}{2\pi} \int_{-\infty}^{\infty}\int \mathbf{E_i^{FT}}\left(\mathbf{k}_{||}, z\right) \mathrm{e}^{\mathrm{j}\left(k_x x + k_y y\right)} \mathrm{d}k_x \mathrm{d}k_y, \tag{5b}$$

thus

$$\mathbf{E_i^{FT}}\left(\mathbf{k}_{||}, z\right) = \mathrm{j}\frac{\omega^2 \mu_1}{4\pi} \frac{1}{k_1^2 k_{z1}} \begin{bmatrix} k_1^2 - k_x^2 & -k_x k_y & \mp k_x k_{z1} \\ -k_x k_y & k_1^2 - k_y^2 & \mp k_y k_{z1} \\ \mp k_x k_{z1} & \mp k_y k_{z1} & k_1^2 - k_{z1}^2 \end{bmatrix} \mathbf{p} \mathrm{e}^{\xi \mathrm{j} k_{z1} z} = \mathrm{j}\frac{\omega^2 \mu_1}{4\pi} \mathbf{M}\left(\mathbf{k}_{||}, z\right) \mathbf{p}, \tag{6}$$

where $\mathbf{k}_{||} = k_x\mathbf{x} + k_y\mathbf{y}$ is the parallel component of $\mathbf{k_1}$ and $\xi = 1$ or -1 for wave propagating in $\mathbf{z}$ or $-\mathbf{z}$ direction, respectively. The electric field can next be written as a sum of two orthogonal polarizations, viz. transverse electric (TE or *s*) and transverse magnetic (TM or *p*). For the electric field to be decomposed, $\mathbf{M}$ must be split into two terms, $\mathbf{M^s}$ (s-polarization) and $\mathbf{M^p}$ (p-polarization), as follows

$$\mathbf{M}\left(\mathbf{k}_{||}, z\right) = \mathbf{M^s}\left(\mathbf{k}_{||}, z\right) + \mathbf{M^p}\left(\mathbf{k}_{||}, z\right), \tag{7}$$

$$\mathbf{M^s}\left(\mathbf{k}_{||}, z\right) = \frac{\mathrm{e}^{\xi \mathrm{j} k_{z1} z}}{k_{z1}\left(k_x^2 + k_y^2\right)} \begin{bmatrix} k_y^2 & -k_x k_y & 0 \\ -k_x k_y & k_x^2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{8}$$

$$\mathbf{M^p}\left(\mathbf{k}_{\|}, z\right)=\frac{\mathrm{e}^{\xi \mathrm{j} k_{z1} z}}{k_1^2\left(k_x^2+k_y^2\right)}\begin{bmatrix} k_x^2 k_{z1} & k_x k_y k_{z1} & -\xi k_x\left(k_x^2+k_y^2\right) \\ k_x k_y k_{z1} & k_y^2 k_{z1} & -\xi k_y\left(k_x^2+k_y^2\right) \\ -\xi k_x\left(k_x^2+k_y^2\right) & -\xi k_y\left(k_x^2+k_y^2\right) & \dfrac{\left(k_x^2+k_y^2\right)^2}{k_{z1}} \end{bmatrix}. \tag{9}$$

In the notation adopted here, the tensors $\mathbf{M^s}$ and $\mathbf{M^p}$ are the outer product of the vectors $\mathbf{L}_{\boldsymbol{\chi}}^{\mathbf{n}}$ and $\mathbf{P}_{\boldsymbol{\chi}}^{\mathbf{n}}$. The first vector is related to the amplitude of $\mathbf{E}_{\mathbf{i}}^{\mathbf{FT}}$, and each vector element corresponds to a dipole polarization, the index n refers to the medium, and χ to the polarization (χ = s for TE or p for TM). The vector $\mathbf{P}_{\boldsymbol{\chi}}^{\mathbf{n}}$ is responsible for decomposing the electric field in x-, y- and z-components. Thus, $\mathbf{M^s}$ and $\mathbf{M^p}$ can be expressed as:

$$\mathbf{M^s}\left(\mathbf{k}_{\|}, z\right)=\mathbf{P_s^n}\left(\mathbf{k}_{\|}\right) \otimes \mathbf{L_s^n}\left(\mathbf{k}_{\|}, z\right)=\begin{bmatrix} \dfrac{k_y}{\sqrt{k_x^2+k_y^2}} \\ -\dfrac{k_x}{\sqrt{k_x^2+k_y^2}} \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \dfrac{k_y}{k_{zn}\sqrt{k_x^2+k_y^2}} e^{\xi \mathrm{j k}_{zn} z} \\ -\dfrac{k_x}{k_{zn}\sqrt{k_x^2+k_y^2}} e^{\xi \mathrm{j k}_{zn} z} \\ 0 \end{bmatrix}, \tag{10}$$

$$\mathbf{M^p}\left(\mathbf{k}_{\|}, z\right)=\mathbf{P_p^n}\left(\mathbf{k}_{\|}\right) \otimes \mathbf{L_p^n}\left(\mathbf{k}_{\|}, z\right)=\begin{bmatrix} \dfrac{k_x k_{zn}}{k_n\sqrt{k_x^2+k_y^2}} \\ \dfrac{k_y k_{zn}}{k_n\sqrt{k_x^2+k_y^2}} \\ -\xi \dfrac{\sqrt{k_x^2+k_y^2}}{k_n} \end{bmatrix} \otimes \begin{bmatrix} \dfrac{k_x}{k_n\sqrt{k_x^2+k_y^2}} e^{\xi \mathrm{j k}_{zn} z} \\ \dfrac{k_y}{k_n\sqrt{k_x^2+k_y^2}} e^{\xi \mathrm{j k}_{zn} z} \\ -\xi \dfrac{\sqrt{k_x^2+k_y^2}}{k_n k_{zn}} e^{\pm \mathrm{j k}_{zn} z} \end{bmatrix}. \tag{11}$$

1.1.1. FT of the Dipole's reflected electric field

The calculation of the dipole's dissipated power requires the knowledge of the total electric field present at the dipole location, which implies that the reflected electric field $\mathbf{E_r}(\mathbf{r})$ needs also to be calculated. According to (4), the dipole's emission is an infinite sum of plane waves. Thus, we first consider a single plane wave with wavevector $\mathbf{k_1^0} = k_x^0\mathbf{x} + k_y^0\mathbf{y} + k_{z1}^0\mathbf{z}$ being radiated by the dipole and impinging on the interface between medium 1 and HMM (z = q). The reflected field FT at the interface ($\mathbf{E_r^{FT,0}}(\mathbf{k}_\|, \mathbf{k}_\|^{\mathbf{0}}, z = q)$ is given by,

$$\begin{aligned} \mathbf{E_r^{FT,0}}\left(\mathbf{k}_{\|}, \mathbf{k}_{\|}^0, q\right)=\mathrm{j} \frac{\omega^2 \mu_1}{4\pi} \sum_{i,m=-\infty}^{\infty}\Big\{&\mathbf{P_s^1}\left(\mathbf{k}_{\|}\right) \otimes\left[\mathbf{L_s^1}\left(\mathbf{k}_{\|}^0, q\right) r_{s,s}^{i,m}\left(\mathbf{k}_{\|}^0\right)+\mathbf{L_p^1}\left(\mathbf{k}_{\|}^0, q\right) r_{p,s}^{i,m}\left(\mathbf{k}_{\|}^0\right)\right] \\ +\mathbf{P_p^1}\left(\mathbf{k}_{\|}\right) \otimes\left[\mathbf{L_s^1}\left(\mathbf{k}_{\|}^0, q\right) r_{s,p}^{i,m}\left(\mathbf{k}_{\|}^0\right)+\mathbf{L_p^1}\left(\mathbf{k}_{\|}^0, q\right) r_{p,p}^{i,m}\left(\mathbf{k}_{\|}^0\right)\right]\Big\} &\mathbf{p}\, \delta\left(k_x-k_x^0-iK_x\right) \delta\left(k_y-k_y^0-mK_y\right) \end{aligned} \tag{12}$$

where $K_x = 2\pi/\Lambda_x$ and $K_y = 2\pi/\Lambda_y$ are the grating momenta along x- and y-directions and $r_{\chi,\sigma}^{i,m}(\mathbf{k}_\|^{\mathbf{0}})$ is the reflection coefficient of both the ith diffraction order in x-direction and the mth diffraction order in y-direction for the parallel incident wavevector $\mathbf{k}_\|^{\mathbf{0}}$. χ and σ indicate the polarization of the incident and reflected waves, respectively (χ and σ are equal to p for TM polarization and s for TE polarization). The reflection coefficient is calculated using the semi-analytical Rigorous Coupled Wave Analysis (RCWA) [2–4] method. For 3D gratings, cross polarization might occur, thus $r_{\chi,\sigma}^{i,m} \neq 0$ for $\chi \neq \sigma$. The next step is to sum all plane waves emitted by the dipole, which is done by integrating $\mathbf{E}_\mathbf{r}^{\mathbf{FT,0}}$ over all possible $\mathbf{k}_\parallel^\mathbf{0}$. Thus, the FT of the reflected electric field emitted by the dipole at the interface $\mathbf{E}_\mathbf{r}^\mathbf{FT}$ is calculated as follows

$$\begin{aligned}\mathbf{E}_\mathbf{r}^\mathbf{FT}\left(\mathbf{k}_\parallel, q\right) &= \iint \mathbf{E}_\mathbf{r}^\mathbf{FT,0}\left(\mathbf{k}_\parallel, \mathbf{k}_\parallel^0, q\right) \mathrm{d}\mathbf{k}_\parallel^0 \\ &= \mathrm{j}\frac{\omega^2\mu_1}{4\pi}\sum_{i,m=-\infty}^{\infty}\left\{\mathbf{P}_\mathbf{s}^\mathbf{1}\left(\mathbf{k}_\parallel\right)\otimes\left[\mathbf{L}_\mathbf{s}^\mathbf{1}\left(\mathbf{k}_\parallel^0, q\right) r_{s,s}^{i,m}\left(\mathbf{k}_\parallel^0\right) + \mathbf{L}_\mathbf{p}^\mathbf{1}\left(\mathbf{k}_\parallel^0, q\right) r_{p,s}^{i,m}\left(\mathbf{k}_\parallel^0\right)\right]\right. \\ &\quad \left. + \mathbf{P}_\mathbf{p}^\mathbf{1}\left(\mathbf{k}_\parallel\right)\otimes\left[\mathbf{L}_\mathbf{s}^\mathbf{1}\left(\mathbf{k}_\parallel^0, q\right) r_{s,p}^{i,m}\left(\mathbf{k}_\parallel^0\right) + \mathbf{L}_\mathbf{p}^\mathbf{1}\left(\mathbf{k}_\parallel^0, q\right) r_{p,p}^{i,m}\left(\mathbf{k}_\parallel^0\right)\right]\right\}\mathbf{p},\end{aligned} \tag{13}$$

$$k_x^0 = k_x - iK_x, \tag{14}$$

$$k_y^0 = k_y - mK_y. \tag{15}$$

1.1.2. FT of the Dipole's transmitted electric field

Similarly, the calculation of the dipole's radiated power to the far field requires calculation of the transmitted electric field $\mathbf{E}_\mathbf{t}(\mathbf{r})$. Using the same approach for $\mathbf{E}_\mathbf{r}^\mathbf{FT}$, the FT of $\mathbf{E}_\mathbf{t}$ at the interface between the grating upper surface and medium 2 ($\mathbf{E}_\mathbf{t}^\mathbf{FT}(\mathbf{k}_\parallel, z = z_{gr})$) is given by

$$\begin{aligned}\mathbf{E}_\mathbf{t}^\mathbf{FT}\left(\mathbf{k}_\parallel, z_{gr}\right) &= \mathrm{j}\frac{\omega^2\mu_1}{4\pi}\sum_{i,m=-\infty}^{\infty}\left\{\mathbf{P}_\mathbf{s}^\mathbf{2}\left(\mathbf{k}_\parallel\right)\otimes\left[\mathbf{L}_\mathbf{s}^\mathbf{1}\left(\mathbf{k}_\parallel^0, q\right) t_{s,s}^{i,m}\left(\mathbf{k}_\parallel^0\right) + \mathbf{L}_\mathbf{p}^\mathbf{1}\left(\mathbf{k}_\parallel^0, q\right) t_{p,s}^{i,m}\left(\mathbf{k}_\parallel^0\right)\right]\right. \\ &\quad \left. + \mathbf{P}_\mathbf{p}^\mathbf{2}\left(\mathbf{k}_\parallel\right)\otimes\left[\mathbf{L}_\mathbf{s}^\mathbf{1}\left(\mathbf{k}_\parallel^0, q\right) t_{s,p}^{i,m}\left(\mathbf{k}_\parallel^0\right) + \mathbf{L}_\mathbf{p}^\mathbf{1}\left(\mathbf{k}_\parallel^0, q\right) t_{p,p}^{i,m}\left(\mathbf{k}_\parallel^0\right)\right]\right\}\mathbf{p},\end{aligned} \tag{16}$$

$$k_x^0 = k_x - iK_x, \tag{17}$$

$$k_y^0 = k_y - mK_y, \tag{18}$$

where $t_{\chi,\sigma}^{i,m}(\mathbf{k}_\parallel^\mathbf{0})$ is the reflection coefficient of the *i*th and *m*th diffraction orders, respective to *x*- and *y*-directions, for the incident wavevector $\mathbf{k}_\mathbf{1}^\mathbf{0}$ with χ and σ as the polarization of the incident and the transmitted fields, respectively. The transmission coefficient is calculated using RCWA. The superscript 2 in $\mathbf{P}_\mathbf{s}^\mathbf{2}(\mathbf{k}_\parallel)$ and $\mathbf{P}_\mathbf{p}^\mathbf{2}(\mathbf{k}_\parallel)$ indicates the electric field is being decomposed in medium 2.

1.1.3. Reflected and transmitted electric fields

The reflected and transmitted electric fields are calculated by inverse Fourier transforming (IFT) their original fields, as follows,

$$\mathbf{E}_\mathbf{r}\left(\mathbf{r}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int \mathbf{E}_\mathbf{r}^\mathbf{FT}\left(\mathbf{k}_\parallel, z\right)\mathrm{e}^{\mathrm{j}\left(k_x x + k_y y\right)}\mathrm{d}k_x\mathrm{d}k_y = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int\left[\mathbf{E}_\mathbf{r}^\mathbf{FT}\left(\mathbf{k}_\parallel, q\right)\mathrm{e}^{-\mathrm{j}k_{z1}\left(z-q\right)}\right]\mathrm{e}^{\mathrm{j}\left(k_x x + k_y y\right)}\mathrm{d}k_x\mathrm{d}k_y, \tag{19}$$

$$\mathbf{E}_\mathbf{t}\left(\mathbf{r}\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int \mathbf{E}_\mathbf{t}^\mathbf{FT}\left(\mathbf{k}_\parallel, z\right)\mathrm{e}^{\mathrm{j}\left(k_x x + k_y y\right)}\mathrm{d}k_x\mathrm{d}k_y = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int\left[\mathbf{E}_\mathbf{t}^\mathbf{FT}\left(\mathbf{k}_\parallel, z_{gr}\right)\mathrm{e}^{\mathrm{j}k_{z1}\left(z-\mathbf{z}_{gr}\right)}\right]\mathrm{e}^{\mathrm{j}\left(k_x x + k_y y\right)}\mathrm{d}k_x\mathrm{d}k_y, \tag{20}$$

The solution proposed in this paper assumes linear media, therefore the total electric field emitted by the dipole is the sum of the electric fields generated by each component of **p**. The solutions are separated in two cases. The first case considers the dipole polarized along the *z*-axis, i.e., perpendicular to the HMM surface, while the second considers the dipole polarized along the *x*- or *y*-axis, i.e., parallel to the HMM surface.

1.2. Perpendicularly polarized dipole

The polarization vector in this case has a component in z-direction, thus $\mathbf{p} = p_z\mathbf{z}$. Substituting $\mathbf{p}$ into (13) and (16), the FT of the reflected and transmitted electric fields become:

$$\mathbf{E}_{\mathbf{r}}^{\mathbf{FT}}\left(\mathbf{k}_{\|}, z\right) = \mathrm{j}\frac{\omega^2\mu}{4\pi} p_z\left[R_{p,s}^z\left(\mathbf{k}_{\|}, q\right)\mathbf{P}_s^1\left(\mathbf{k}_{\|}\right) + R_{p,p}^z\left(\mathbf{k}_{\|}, q\right)\mathbf{P}_{\mathbf{p}}^1\left(\mathbf{k}_{\|}\right)\right]\mathrm{e}^{-\mathrm{j}k_{z1}(z-q)}, \tag{21}$$

$$\mathbf{E}_{\mathbf{t}}^{\mathbf{FT}}\left(\mathbf{k}_{\|}, z\right) = \mathrm{j}\frac{\omega^2\mu}{4\pi} p_z\left[T_{p,s}^z\left(\mathbf{k}_{\|}, z_{gr}\right)\mathbf{P}_s^1\left(\mathbf{k}_{\|}\right) + T_{p,p}^z\left(\mathbf{k}_{\|}, z_{gr}\right)\mathbf{P}_{\mathbf{p}}^1\left(\mathbf{k}_{\|}\right)\right]\mathrm{e}^{\mathrm{j}k_{z1}(z-z_{gr})}, \tag{22}$$

$$R_{p,\sigma}^z\left(\mathbf{k}_{\|}, q\right) = -\sum_{i,m=-\infty}^{\infty}\frac{\sqrt{\left(k_x^0\right)^2+\left(k_y^0\right)^2}}{k_{z1}^0 k_1}e^{\mathrm{j}k_{z1}^0 q} r_{p,\sigma}^{i,m}\left(\mathbf{k}_{\|}^{\mathbf{0}}\right), \tag{23}$$

$$T_{p,\sigma}^z\left(\mathbf{k}_{\|}, z_{gr}\right) = -\sum_{i,m=-\infty}^{\infty}\frac{\sqrt{\left(k_x^0\right)^2+\left(k_y^0\right)^2}}{k_{z1}^0 k_1}\mathrm{e}^{\mathrm{j}k_{z1}^0 q} t_{p,\sigma}^{i,m}\left(\mathbf{k}_{\|}^{\mathbf{0}}\right). \tag{24}$$

With the reflected and transmitted electric fields calculated, the next step is to obtain the Purcell factor and the dipole's total radiated power into medium 2, as described next.

1.2.1. Purcell factor

The first step is to determine the dipole's total dissipated power (W), which is carried out as follows:

$$W = \frac{\omega}{2}\mathrm{Im}\left\{\mathbf{p}^*\cdot\mathbf{E}_{\mathbf{i}}\left(\mathbf{r}_{\mathbf{0}}\right) + \mathbf{p}^*\cdot\mathbf{E}_{\mathbf{r}}\left(\mathbf{r}_{\mathbf{0}}\right)\right\}, \tag{25}$$

$$\begin{aligned} W &= \frac{\omega}{2}\mathrm{Im}\left\{\frac{1}{2\pi}\int_{-\infty}^{\infty}\int\left[\mathrm{j}\frac{\omega^2\mu_1}{4\pi}p_z^2\frac{k_x^2+k_y^2}{k_1^2 k_{z1}} + \mathrm{j}\frac{\omega^2\mu_1}{4\pi}p_z^2\left(R_{p,s}^z\left(\mathbf{k}_{\|}, q\right)\mathbf{P}_s^1\left(\mathbf{k}_{\|}\right) + R_{p,p}^z\left(\mathbf{k}_{\|}, q\right)\mathbf{P}_{\mathbf{p}}^1\left(\mathbf{k}_{\|}\right)\right)\cdot\mathbf{z}\mathrm{e}^{\mathrm{j}k_{z1}q}\right]\mathrm{d}k_x\mathrm{d}k_y\right\} \\ &= \frac{\omega^3\mu_1}{16\pi^2}p_z^2\,\mathrm{Re}\left\{\int_{-\infty}^{\infty}\int\left[\frac{k_x^2+k_y^2}{k_1^2 k_{z1}} + \left(R_{p,s}^z\left(\mathbf{k}_{\|}, q\right)\mathbf{P}_s^1\left(\mathbf{k}_{\|}\right) + R_{p,p}^z\left(\mathbf{k}_{\|}, q\right)\mathbf{P}_{\mathbf{p}}^1\left(\mathbf{k}_{\|}\right)\right)\cdot\mathbf{z}e^{\mathrm{j}k_{z1}q}\right]\mathrm{d}k_x\mathrm{d}k_y\right\}. \end{aligned} \tag{26}$$

The Purcell factor is defined as the ratio of the dipole's dissipated power to that it would dissipate in freespace ($W_0 = p_z^2\omega k_0^3/12\pi\epsilon_0$). Since $\mathbf{P}_{\mathbf{s}}^{\mathbf{1}}(\mathbf{k}_{\|})\cdot\mathbf{z} = 0$, the Purcell factor for perpendicular polarization (P_z) is given by:

$$P_z = \frac{W}{W_0} = \frac{k_1}{k_0} + \frac{3}{4\pi k_0}\mathrm{Re}\left\{\int_{-\infty}^{\infty}\int\left[R_{p,p}^z\left(\mathbf{k}_{\|}, q\right)\mathbf{P}_{\mathbf{p}}^1\left(\mathbf{k}_{\|}\right)\right]\cdot\mathbf{z}e^{\mathrm{j}k_{z1}q}\mathrm{d}k_x\mathrm{d}k_y\right\}. \tag{27}$$

1.2.2. Radiated power

Our main goal is to be able to increase the coupling of evanescent waves inside the HMM to propagating waves in the medium 2. In this sense, it is important to calculate the dipole's total emitted power into the far field in medium 2. Using the stationary phase method [1], the transmitted electric field at the far field ($\mathbf{E}_{\mathbf{z}}^{\mathbf{FF}}$) is calculated as follows,

$$
\begin{aligned}
\mathbf{E}_z^{\mathbf{FF}}(\mathbf{r}) &= -\mathrm{j}k_{z2}\mathbf{E}_\mathbf{t}^{\mathbf{FT}}\left(\mathbf{k}_{\|},0\right)\frac{e^{\mathrm{j}|\mathbf{k_2}|r}}{r} \\
&= \frac{\omega^2\mu_1|\mathbf{k_2}|}{4\pi}p_z\cos\theta\left[T_{p,s}^z\left(\mathbf{k}_{\|},z_{gr}\right)\boldsymbol{\varphi}+T_{p,p}^z\left(\mathbf{k}_{\|},z_{gr}\right)\boldsymbol{\theta}\right]\frac{e^{\mathrm{j}|\mathbf{k_2}|r}}{r}e^{-\mathrm{j}|\mathbf{k_2}|\cos\theta z_{gr}},
\end{aligned}
\tag{28}
$$

where r, θ and φ are the components of $\mathbf{r}$ in spherical coordinates. For the far-field equation to be used in (28) the following transformation is required:

$$
\mathbf{k}_2=\begin{bmatrix}k_x\\k_y\\k_{z2}\end{bmatrix}=k_2\begin{bmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{bmatrix},\ \begin{matrix}0\le\theta\le\pi/2\\0\le\varphi\le2\pi\end{matrix}.
\tag{29}
$$

The transformation of $\mathbf{E_z^{FF}}(\mathbf{r})$ into spherical coordinates is carried out after substituting (29) into (10) and (11), which gives

$$
\mathbf{P_p^2}\left(\mathbf{k}_{\|}\right)=\frac{k_xk_{z2}}{k_2\sqrt{k_x^2+k_y^2}}\mathbf{x}+\frac{k_yk_{z2}}{k_2\sqrt{k_x^2+k_y^2}}\mathbf{y}-\frac{\sqrt{k_x^2+k_y^2}}{k_2}\mathbf{z}=\boldsymbol{\theta},
\tag{30}
$$

$$
\mathbf{P_s^2}\left(\mathbf{k}_{\|}\right)=\frac{k_y}{\sqrt{k_x^2+k_y^2}}\mathbf{x}-\frac{k_x}{\sqrt{k_x^2+k_y^2}}\mathbf{y}=\boldsymbol{\varphi}.
\tag{31}
$$

With $\mathbf{E_z^{FF}}$ known, the time-averaged Poynting vector ($\langle \boldsymbol{S_z}(\mathbf{r})\rangle$) is calculated as follows,

$$
\left\langle\mathbf{S_z}(\mathbf{r})\right\rangle=\frac{1}{2z_2}\mathrm{Re}\left\{\mathbf{E}_z^{\mathbf{FF}}\cdot\mathbf{E}_z^{\mathbf{FF}*}\right\}\mathbf{r}=\frac{\omega^4\mu_1^2k_2}{32\pi^2z_2r^2}p_z^2\cos^2\theta\left(\left|T_{p,s}^z\left(\mathbf{k}_{\|},z_{gr}\right)\right|^2+\left|T_{p,p}^z\left(\mathbf{k}_{\|},z_{gr}\right)\right|^2\right)\mathbf{r}.
\tag{32}
$$

The total radiated power into the far-field (Q_z) of a perpendicularly polarized dipole is calculated integrating $\langle\mathbf{S_z}\rangle$ over the upper semi-sphere, i.e.

$$
Q_z=\int_0^{\pi/2}\int_0^{2\pi}\left\langle\mathbf{S_z}\right\rangle\cdot\mathbf{r}\sin\theta r^2\mathrm{d}\varphi\mathrm{d}\theta=\frac{\omega^4\mu_1^2k_2p_z^2}{32\pi^2z_2}\int_0^{\pi/2}\int_0^{2\pi}\cos^2\theta\sin\theta\left(\left|T_{p,s}^z\left(\mathbf{k}_{\|},z_{gr}\right)\right|^2+\left|T_{p,p}^z\left(\mathbf{k}_{\|},z_{gr}\right)\right|^2\right)\mathrm{d}\varphi\mathrm{d}\theta.
\tag{33}
$$

1.3. Parallel polarized dipole

The polarization vector in this case is oriented along the x-axis, resulting in $\mathbf{p} = p_x\mathbf{x}$. The equations derived in this section can be easily adapted to the y-axis by simply changing the terms k_x to k_y, and k_y to k_x. After substituting $\mathbf{p}$ into (13) and (16), the following expressions for the FTs of the reflected and transmitted electric fields are obtained:

$$
\begin{aligned}
\mathbf{E_r^{FT}}\left(\mathbf{k}_{\|},z\right)=\mathrm{j}\frac{\omega^2\mu}{4\pi}p_x&\left\{\left[R_{s,s}^x\left(\mathbf{k}_{\|},q\right)+R_{p,s}^x\left(\mathbf{k}_{\|},q\right)\right]\mathbf{P_s^1}\left(\mathbf{k}_{\|}\right)\right.\\
&\left.+\left[R_{p,p}^x\left(\mathbf{k}_{\|},q\right)+R_{s,p}^x\left(\mathbf{k}_{\|},q\right)\right]\mathbf{P_p^1}\left(\mathbf{k}_{\|}\right)\right\}e^{-\mathrm{j}k_{z1}(z-q)}
\end{aligned}
\tag{34}
$$

$$
\begin{aligned}
\mathbf{E_t^{FT}}\left(\mathbf{k}_{\|},z\right)=\mathrm{j}\frac{\omega^2\mu}{4\pi}p_x&\left\{\left[T_{s,s}^x\left(\mathbf{k}_{\|},z_{gr}\right)+T_{p,s}^x\left(\mathbf{k}_{\|},z_{gr}\right)\right]\mathbf{P_s^2}\left(\mathbf{k}_{\|}\right)\right.\\
&\left.+\left[T_{p,p}^x\left(\mathbf{k}_{\|},z_{gr}\right)+T_{s,p}^x\left(\mathbf{k}_{\|},z_{gr}\right)\right]\mathbf{P_p^2}\left(\mathbf{k}_{\|}\right)\right\}e^{\mathrm{j}k_{z1}\left(z-z_{gr}\right)},
\end{aligned}
\tag{35}
$$

$$R_{s,\sigma}^{x}\left(\mathbf{k}_{\|},q\right)=\sum_{i,m=-\infty}^{\infty}\frac{k_{y}^{0}}{k_{z1}^{0}\sqrt{\left(k_{x}^{0}\right)^{2}+\left(k_{y}^{0}\right)^{2}}}e^{\mathrm{j}k_{z1}^{0}q}r_{s,\sigma}^{i,m}\left(\mathbf{k}_{\|}^{\mathbf{0}}\right),\tag{36}$$

$$R_{p,\sigma}^{x}\left(\mathbf{k}_{\|},q\right)=\sum_{i,m=-\infty}^{\infty}\frac{k_{x}^{0}}{k_{1}^{0}\sqrt{\left(k_{x}^{0}\right)^{2}+\left(k_{y}^{0}\right)^{2}}}e^{\mathrm{j}k_{z1}^{0}q}r_{p,\sigma}^{i,m}\left(\mathbf{k}_{\|}^{\mathbf{0}}\right),\tag{37}$$

$$T_{s,\sigma}^{x}\left(\mathbf{k}_{\|},z_{gr}\right)=\sum_{i,m=-\infty}^{\infty}\frac{k_{y}^{0}}{k_{z1}^{0}\sqrt{\left(k_{x}^{0}\right)^{2}+\left(k_{y}^{0}\right)^{2}}}e^{\mathrm{j}k_{z1}^{0}q}t_{s,\sigma}^{i,m}\left(\mathbf{k}_{\|}^{\mathbf{0}}\right),\tag{38}$$

$$T_{p,\sigma}^{x}\left(\mathbf{k}_{\|},z_{gr}\right)=\sum_{i,m=-\infty}^{\infty}\frac{k_{x}^{0}}{k_{1}^{0}\sqrt{\left(k_{x}^{0}\right)^{2}+\left(k_{y}^{0}\right)^{2}}}e^{\mathrm{j}k_{z1}^{0}q}t_{p,\sigma}^{i,m}\left(\mathbf{k}_{\|}^{\mathbf{0}}\right).\tag{39}$$

1.3.1. Purcell factor

The Purcell factor for parallel polarization is obtained after substituting (34) into (25) and the resulting equation into (27), which gives:

$$\begin{aligned}P_{x}=\frac{k_{1}}{k_{0}}+\frac{3}{4\pi k_{0}}\mathrm{Re}\Bigg\{\int_{-\infty}^{\infty}\int\Big[\Big(R_{s,s}^{x}\left(\mathbf{k}_{\|},q\right)+R_{p,s}^{x}\left(\mathbf{k}_{\|},q\right)\Big)\mathbf{P}_{s}^{1}\left(\mathbf{k}_{\|}\right)+\\+\Big(R_{p,p}^{x}\left(\mathbf{k}_{\|},q\right)R_{s,p}^{x}\left(\mathbf{k}_{\|},q\right)\Big)\mathbf{P}_{\mathbf{p}}^{1}\left(\mathbf{k}_{\|}\right)\Big]\cdot\mathbf{x}e^{\mathrm{j}k_{z1}q}\mathrm{d}k_{x}\mathrm{d}k_{y}\Bigg\}.\end{aligned}\tag{40}$$

1.3.2. Radiated power

The total dipole’s radiated power into the far field requires the electric field calculation in the far field ($\mathbf{E}_{\mathbf{x}}^{\mathbf{FF}}$). As in section 1.2.2, this is accomplished with the stationary phase method [1] as follows,

$$\begin{aligned}\mathbf{E}_{x}^{\mathbf{FF}}\left(\mathbf{r}\right)&=-\mathrm{j}k_{z2}\mathbf{E}_{\mathbf{t}}^{\mathbf{FT}}\left(\mathbf{k}_{\|},0\right)\frac{e^{\mathrm{j}k_{2}r}}{r}\\&=\frac{\omega^{2}\mu_{1}k_{2}}{4\pi}p_{x}\cos\theta\Big\{\Big[T_{s,s}^{x}\left(\mathbf{k}_{\|},z_{gr}\right)+T_{p,s}^{x}\left(\mathbf{k}_{\|},z_{gr}\right)\Big]\boldsymbol{\varphi}+\Big[T_{s,p}^{x}\left(\mathbf{k}_{\|},z_{gr}\right)+T_{p,p}^{x}\left(\mathbf{k}_{\|},z_{gr}\right)\Big]\boldsymbol{\theta}\Big\}\frac{e^{\mathrm{j}k_{2}r}}{r}e^{-\mathrm{j}k_{2}\cos\theta z_{gr}}.\end{aligned}\tag{41}$$

The far field approximation in (41) also makes use of (29). Equations (30) and (31) transform $\mathbf{E}_{\mathbf{x}}^{\mathbf{FF}}$ into spherical coordinates, from which the time-averaged Poynting vector ($\langle\mathbf{S}_{\mathbf{x}}\rangle$) is calculated as follows,

$$\left\langle\mathbf{S}_{\mathbf{x}}\left(\mathbf{r}\right)\right\rangle=\frac{1}{2z_{2}}\mathrm{Re}\left\{\mathbf{E}_{\mathbf{t}}^{\mathbf{FF}}\cdot\mathbf{E}_{\mathbf{t}}^{\mathbf{FF}*}\right\}\mathbf{r}=\frac{\omega^{4}\mu_{1}^{2}k_{2}}{32\pi^{2}z_{2}r^{2}}p_{x}^{2}\cos^{2}\theta\left(\left|T_{s,s}^{x}\left(\mathbf{k}_{\|},z_{gr}\right)+T_{p,s}^{x}\left(\mathbf{k}_{\|},z_{gr}\right)\right|^{2}+\left|T_{s,p}^{x}\left(\mathbf{k}_{\|},z_{gr}\right)+T_{p,p}^{x}\left(\mathbf{k}_{\|},z_{gr}\right)\right|^{2}\right)\mathbf{r}.\tag{42}$$

The total radiated power into the far field (Q_x) by a parallel polarized dipole is calculated using the same approach adopted for the perpendicularly polarized case, which gives,

$$\begin{aligned}Q_{x}=\int_{0}^{\pi/2}\int_{0}^{2\pi}\left\langle\mathbf{S}_{\mathbf{x}}\right\rangle\cdot\mathbf{r}r^{2}\sin\theta\mathrm{d}\varphi\mathrm{d}\theta=\frac{\omega^{4}\mu_{1}^{2}k_{2}p_{x}^{2}}{32\pi^{2}z_{2}}\int_{0}^{\pi/2}\int_{0}^{2\pi}\cos^{2}\theta\sin\theta\Bigg(\left|T_{s,s}^{x}\left(\mathbf{k}_{\|},z_{gr}\right)+T_{p,s}^{x}\left(\mathbf{k}_{\|},z_{gr}\right)\right|^{2}\\+\left|T_{s,p}^{x}\left(\mathbf{k}_{\|},z_{gr}\right)+T_{p,p}^{x}\left(\mathbf{k}_{\|},z_{gr}\right)\right|^{2}\Bigg)\mathrm{d}\varphi\mathrm{d}\theta.\end{aligned}\tag{43}$$

2. 2D Configuration

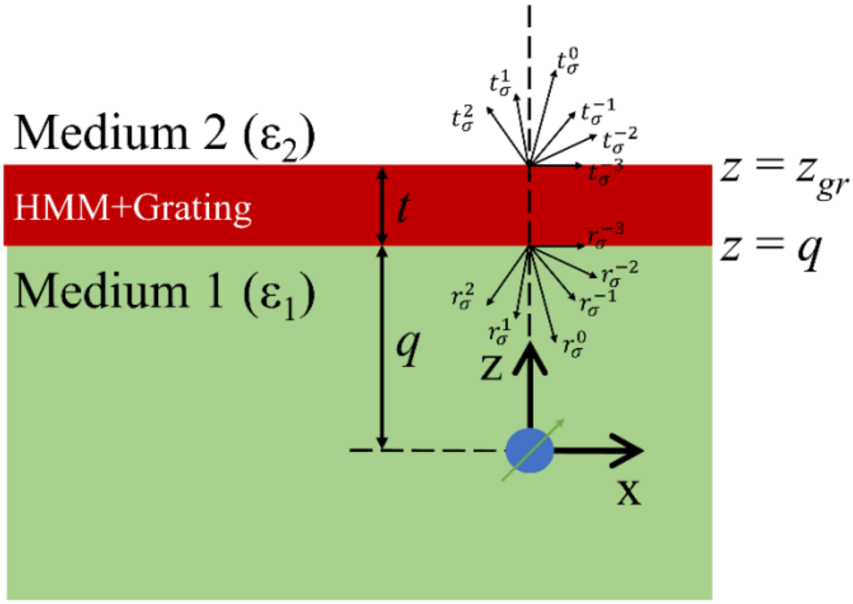

Figure S2. Proposed 2D geometry in which the QEs are modelled as dipoles embedded in medium 1 (with permittivity ε_1), centered at the origin and at a distance q from the HMM bottom surface. The grating with periodicity Λ_x in x-direction is on top of the HMM and is covered with medium 2 (with permittivity ε_2). The HMM and the grating are merged for sake of clarity.

For the 2D case we consider the dipole embedded in the medium 1 (with permittivity ε_1) at a distance q from the HMM bottom surface, as shown in Fig. S2. The HMM and the grating are merged in this figure for sake of clarity. The grating is surrounded by medium 2 (with permittivity ε_2) and has periodicity Λ_x along the x-axis. The most significant difference between the 3D and 2D approaches is that the system is constant along the y-axis, i.e. $\partial/\partial y = 0$, resulting in $k_y = 0$. Therefore, the procedure to calculate the Purcell factor and the radiated power is different from the 3D scenario and starts with the calculation of the transmitted and reflected electric fields.

2.1. Calculation of the transmitted and reflected electric fields

We start by calculating the electric field radiated by the dipole in medium 1, which requires the dipole to be modelled as the following current source in 2D,

$$\mathbf{J}(\mathbf{r}) = -\mathrm{j}\omega\delta^{(2)}(x,z)\mathbf{p} \tag{44}$$

where $\mathbf{p}$ is the dipole momentum and $\delta^{(2)}(x,z)$ is the 2D Dirac delta function. The electric field radiated by the dipole ($\mathbf{E}_\mathbf{i}$) in 2D becomes

$$\mathbf{E}_\mathbf{i}(\boldsymbol{\rho}) = \omega^2\mu_1\iint_{A'}\left(\mathbf{I}+\frac{1}{k_1^2}\nabla\nabla\right)\mathrm{G}_0(\boldsymbol{\rho}-\boldsymbol{\rho}')\delta^{(2)}(x',z')\mathbf{p}\,\mathrm{d}A', \tag{45}$$

$$\mathrm{G}_0(\boldsymbol{\rho}) = \frac{\mathrm{j}}{4}H_0^1(\mathbf{k_1}\boldsymbol{\rho}) = \mathrm{j}\frac{1}{4\pi}\int_{-\infty}^{\infty}\frac{1}{\sqrt{k_0^2-k_x^2}}\mathrm{e}^{\mathrm{j}(k_x x+\xi k_{z1}z)}\mathrm{d}k_x \tag{46}$$

$$\boldsymbol{\rho} = x\mathbf{x} + z\mathbf{z}, \tag{47}$$

where H_0^1 is the first Hankel function of order 0, G_0 is the scalar Green's function in 2D [5] and $\boldsymbol{\rho}$ is the position vector in 2D. Substituting (46) into (45), the following expression for $\mathbf{E}_\mathbf{i}(\boldsymbol{\rho})$ is obtained

$$\mathbf{E}_{\mathbf{i}}(\boldsymbol{\rho}) = \mathrm{j}\frac{\omega^2\mu_1}{4\pi}\left(\mathbf{I} + \frac{1}{k_1^2}\nabla\nabla\right)\int_{-\infty}^{\infty}\frac{1}{k_{z1}}\mathrm{e}^{\mathrm{j}(k_x x + \xi k_{z1} z)}\mathbf{p}\mathrm{d}k_x. \tag{48}$$

By considering the 2D FT and the inverse FT defined as

$$\mathbf{E}_{\mathbf{i}}^{\mathbf{FT}}(k_x, z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{E}_{\mathbf{i}}(\mathbf{r})\mathrm{e}^{-\mathrm{j}k_x x}\mathrm{d}x, \tag{49a}$$

$$\mathbf{E}_{\mathbf{i}}(\boldsymbol{\rho}) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{E}_{\mathbf{i}}^{\mathbf{FT}}(k_x, z)\mathrm{e}^{\mathrm{j}(k_x x)}\mathrm{d}k_x, \tag{49b}$$

we obtain

$$\mathbf{E}_{\mathbf{i}}^{\mathbf{FT}}(k_x, z) = \mathrm{j}\frac{\omega^2\mu_1}{\sqrt{8\pi}}\frac{1}{k_1^2 k_{z1}}\begin{bmatrix} k_1^2 - k_x^2 & 0 & -\xi k_x k_{z1} \\ 0 & k_1^2 & 0 \\ -\xi k_x k_{z1} & 0 & k_1^2 - k_{z1}^2 \end{bmatrix}\mathbf{p}\mathrm{e}^{\mathrm{j}\xi k_{z1} z} = \mathrm{j}\frac{\omega^2\mu_1}{\sqrt{8\pi}}\mathbf{M}(k_x, z)\mathbf{p}. \tag{50}$$

In 2D, the dyadic tensor **M** has TE and TM components and can be written as the outer product of two vectors, $\mathbf{L}_{\boldsymbol{\chi}}^{\mathbf{n}}$ and $\mathbf{P}_{\boldsymbol{\chi}}^{\mathbf{n}}$, whose physical meaning is the same as for the 3D case. Thus, **M** is written as:

$$\mathbf{M}(\mathbf{k}_{\parallel}, z) = \mathbf{M}^{\mathrm{s}}(\mathbf{k}_{\parallel}, z) + \mathbf{M}^{\mathrm{p}}(\mathbf{k}_{\parallel}, z), \tag{51a}$$

$$\mathbf{M}^{\mathrm{s}}(k_x, z) = \mathbf{P}_{\mathbf{s}}^{\mathbf{1}}(k_x) \otimes \mathbf{L}_{\mathbf{s}}^{\mathbf{1}}(k_x, z) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ \dfrac{1}{k_{z1}}\mathrm{e}^{\xi \mathrm{j} k_{z1} z} \\ 0 \end{bmatrix}. \tag{51b}$$

$$\mathbf{M}^{\mathrm{p}}(k_x, z) = \mathbf{P}_{\mathbf{p}}^{\mathbf{1}}(k_x) \otimes \mathbf{L}_{\mathbf{p}}^{\mathbf{1}}(k_x, z) = \begin{bmatrix} \dfrac{k_{z1}}{k_1} \\ 0 \\ -\xi\dfrac{k_x}{k_1} \end{bmatrix} \otimes \begin{bmatrix} \dfrac{1}{k_1}\mathrm{e}^{\xi \mathrm{j} k_{z1} z} \\ 0 \\ -\xi\dfrac{k_x}{k_1 k_{z1}}\mathrm{e}^{\xi \mathrm{j} k_{z1} z} \end{bmatrix}. \tag{51c}$$

The dipole only generates TE waves if polarized along y-axis. Nevertheless, if **p** is in the xz-plane, only TM waves are generated. Since we are working with hyperbolic metamaterials, which only couples TM waves, for the 2D case, we will consider **p** in xz-plane.

2.1.1. FT of the Dipole's reflected electric field

The calculation of the reflected electric field is based on the same approach used for the 3D case. Firstly, we consider a single plane wave with wavevector $\mathbf{k_1^0} = k_x^0\mathbf{x} + k_{z1}^0\mathbf{z}$ impinging on the interface between medium 1 and HMM (*z*=*q*). The FT of the reflected field ($\mathbf{E}_{\mathbf{r}}^{\mathbf{FT,0}}(k_x, k_x^0, q)$) of this wave at *z*=*q* is given by,

$$\mathbf{E}_{\mathbf{r}}^{\mathbf{FT,0}}(k_x, k_x^0, q) = \mathrm{j}\frac{\omega^2\mu_1}{\sqrt{8\pi}}\sum_{i=-\infty}^{\infty}\left\{\mathbf{P}_{\mathbf{p}}^{\mathbf{1}}(k_x) \otimes \left[\mathbf{L}_{\mathbf{p}}^{\mathbf{1}}(k_x^0, q) r_p^i(k_x^0)\right]\right\}\mathbf{p}\delta(k_x - k_x^0 - iK_x) \tag{52}$$

where $K_x = 2\pi/\Lambda_x$ is the grating momentum in *x*-direction and r_p^i is the reflection coefficient of the *i*th diffraction order of a *p*-polarized wave. In 2D, all dipoles radiate *p*-polarized waves, therefore the magnetic field is oriented along the y-axis. Since the grating changes the wave momentum in the xz plane, only the electric field vector, which is in the xz plane, changes its direction. Therefore, the wave remains p-polarized and no cross-polarization coefficients appear in (52). Thus, the FT of the reflected electric field radiated by the dipole $\mathbf{E}_\mathbf{r}^\mathbf{FT}$ can be calculated integrating $\mathbf{E}_\mathbf{r}^\mathbf{FT,0}$ over all possible k_x^0, i.e.,

$$\begin{aligned}\mathbf{E}_\mathbf{r}^\mathbf{FT}\left(k_x,q\right) &= \int_{-\infty}^{\infty}\mathbf{E}_\mathbf{r}^\mathbf{FT,0}\left(k_x,k_x^0,q\right)\mathrm{d}k_x^0\\ &= \int_{-\infty}^{\infty}\mathrm{j}\frac{\omega^2\mu_1}{\sqrt{8\pi}}\sum_i\left\{\mathbf{P}_\mathbf{p}^\mathbf{1}\left(k_x\right)\otimes\left[\mathbf{L}_\mathbf{p}^\mathbf{1}\left(k_x^0,q\right)r_p^i\left(k_x^0\right)\right]\right\}\mathbf{p}\,\delta\left(k_x-k_x^0-iK_x\right)\mathrm{d}k_x^0\\ &= \mathrm{j}\frac{\omega^2\mu_1}{\sqrt{8\pi}}\sum_i\left\{\mathbf{P}_\mathbf{p}^\mathbf{1}\left(k_x\right)\otimes\left[\mathbf{L}_\mathbf{p}^\mathbf{1}\left(k_x^0,q\right)r_p^i\left(k_x^0\right)\right]\right\}\mathbf{p},\end{aligned} \tag{53}$$

$$k_x^0 = k_x - iK_x. \tag{54}$$

2.1.1. *FT of the Dipole's transmitted electric field*

The FT of the transmitted electric field ($\mathbf{E}_\mathbf{t}^\mathbf{FT}$) follows the same methodology adopted for the reflected field. Thus, at the interface between the grating upper surface and medium 2 ($z = z_{gr}$) we obtain,

$$\mathbf{E}_\mathbf{t}^\mathbf{FT}\left(k_x,z_{gr}\right) = \mathrm{j}\frac{\omega^2\mu_1}{\sqrt{8\pi}}\sum_i\left\{\mathbf{P}_\mathbf{p}^2\left(k_x\right)\otimes\left[\mathbf{L}_\mathbf{p}^\mathbf{1}\left(k_x^0,q\right)t_p^i\left(k_x^0\right)\right]\right\}\mathbf{p}. \tag{55}$$

The superscript 2 in $\mathbf{P}_\mathbf{p}^\mathbf{2}(k_x)$ indicates the electric field is being decomposed in medium 2.

2.1.2. *Reflected and transmitted electric fields*

Since we have the FT of the transmitted and reflected fields, the electric fields in the spatial domain are obtained using the IFT, as follows,

$$\mathbf{E}_\mathbf{r}\left(\boldsymbol{\rho}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{E}_\mathbf{r}^\mathbf{FT}\left(k_x,z\right)\mathrm{e}^{\mathrm{j}k_x x}\mathrm{d}k_x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\mathbf{E}_\mathbf{r}^\mathbf{FT}\left(k_x,q\right)\mathrm{e}^{-\mathrm{j}k_{z1}\left(z-q\right)}\right]\mathrm{e}^{\mathrm{j}k_x x}\mathrm{d}k_x, \tag{56}$$

$$\mathbf{E}_\mathbf{t}\left(\boldsymbol{\rho}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\mathbf{E}_\mathbf{t}^\mathbf{FT}\left(k_x,z\right)\mathrm{e}^{\mathrm{j}k_x x}\mathrm{d}k_x = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left[\mathbf{E}_\mathbf{t}^\mathbf{FT}\left(k_x,q\right)\mathrm{e}^{\mathrm{j}k_{z1}\left(z-z_{gr}\right)}\right]\mathrm{e}^{\mathrm{j}k_x x}\mathrm{d}k_x, \tag{57}$$

In 2D, **p** can be decomposed in x and z components, therefore the solutions hereon are separated into two cases. The first considers the dipole polarized along the z-axis, i.e., perpendicular to the HMM surface, while the second considers the dipole polarized along the x-axis, i.e., parallel to the HMM surface.

2.2. Perpendicularly polarized dipole

The dipole moment in this case is polarized along the z-axis, thus $\mathbf{p} = p_z\mathbf{z}$. After substituting **p** into (53) and (55), the following FTs of the reflected and transmitted field electric are obtained:

$$\mathbf{E}_\mathbf{r}^\mathbf{FT}\left(k_x,z\right) = \mathrm{j}\frac{\omega^2\mu_1}{\sqrt{8\pi}}p_z R_p^z\left(k_x,q\right)\mathbf{P}_\mathbf{p}^\mathbf{1}\left(k_x\right)e^{-\mathrm{j}k_{z1}\left(z-q\right)}, \tag{58}$$

$$\mathbf{E}_\mathbf{t}^\mathbf{FT}\left(k_x,z\right) = \mathrm{j}\frac{\omega^2\mu_1}{\sqrt{8\pi}}p_z T_p^z\left(k_x,z_{gr}\right)\mathbf{P}_\mathbf{p}^2\left(k_x\right)e^{\mathrm{j}k_{z1}\left(z-z_{gr}\right)}, \tag{59}$$

$$R_p^z\left(k_x,q\right) = -\sum_{i=-\infty}^{\infty}\frac{k_x^0}{k_{z1}^0k_1}e^{\mathrm{j}k_{z1}^0q}r_p^i\left(k_x^0\right), \tag{60}$$

$$T_p^z\left(k_x, z_{gr}\right) = -\sum_{i=-\infty}^{\infty} \frac{k_x^0}{k_{z1}^0 k_1} e^{\mathrm{j}k_{z1}^0 q} t_p^i\left(k_x^0\right). \tag{61}$$

2.2.1. Purcell factor

Before calculating the Purcell factor, we need to define the energy that the dipole would dissipate in free space (W_0), which is calculated as follows,

$$W_0 = \frac{\omega}{2}\mathrm{Im}\left\{\mathbf{p}^* \cdot \mathbf{E}_\mathrm{i}\left(\mathbf{0}\right)\right\} = \frac{\omega}{2}\frac{\omega^2 \mu_1}{\sqrt{8\pi}} p_z^2 \frac{1}{\sqrt{2\pi}} \mathrm{Re}\left\{\int_{-\infty}^{\infty} \frac{k_x^2}{k_{z0}k_0^2}\mathrm{d}k_x\right\} = \frac{\omega^3\mu_1}{16} p_z^2. \tag{62}$$

The next step consists in calculating the total power dissipated by the dipole under the proposed structure, which is accomplished by substituting (58) into (56) and then into (25), which gives

$$W = \frac{\omega^3\mu_1}{8\pi} p_z^2 \,\mathrm{Re}\left\{\int_{-\infty}^{\infty}\left[\frac{k_x^2}{k_{z1}k_1^2} + R_p^z\left(k_x, q\right)\left(\mathbf{P}_\mathbf{p}^\mathbf{1}\left(k_x\right)\cdot \mathbf{z}\right)\mathrm{e}^{\mathrm{j}k_{z1}q}\right]\mathrm{d}k_x\right\}. \tag{63}$$

Therefore, the Purcell factor for a 2D perpendicularly polarized dipole (P_z^{2D}) is calculated as follows,

$$P_z^{2D} = \frac{W}{W_0} = 1 + \frac{2}{\pi}\mathrm{Re}\left\{\int_{-\infty}^{\infty} R_p^z\left(k_x, q\right)\left[\mathbf{P}_\mathbf{p}^\mathbf{1}\left(k_x\right)\cdot\mathbf{z}\right]\mathrm{e}^{\mathrm{j}k_{z1}q}\mathrm{d}k_x\right\}. \tag{64}$$

2.2.2. Radiated Power

The dipole's total radiated power can be calculated by integrating the far field time averaged Poynting vector over the upper semi-circle. To this end, we need to calculate the electric field generated by a perpendicularly polarized dipole in the far field ($\mathbf{E}_\mathbf{z}^\mathbf{FF}$), which according to the stationary phase method is written as follows,

$$\mathbf{E}_\mathbf{z}^\mathbf{FF}\left(\mathbf{r}\right) = -\mathrm{j}\frac{k_{z2}}{\sqrt{k_2}} E_t^{FT}\left(k_x, 0\right)\frac{\mathrm{e}^{\mathrm{j}k_2|\boldsymbol{\rho}|}}{\sqrt{|\boldsymbol{\rho}|}}\mathrm{e}^{\text{-}\mathrm{j}k_{z2}z_{gr}} = \frac{\omega^2\mu_1}{\sqrt{8\pi}} p_z \sqrt{k_2}\cos\theta T_p^z \frac{\mathrm{e}^{\mathrm{j}k_2|\boldsymbol{\rho}|}}{\sqrt{|\boldsymbol{\rho}|}}\mathrm{e}^{\text{-}\mathrm{j}k_2\cos\theta z_{gr}}\boldsymbol{\theta}, \tag{65}$$

where ρ and θ are the components of $\boldsymbol{\rho}$ in cylindrical coordinates (in the *xz* plane). The following has been assumed for the far-field approximation in (65),

$$\mathbf{k}_2 = \begin{bmatrix} k_x \\ 0 \\ k_{z2}\end{bmatrix} = k_2\begin{bmatrix}\sin\theta \\ 0 \\ \cos\theta\end{bmatrix}, -\pi/2 \le \theta \le \pi/2. \tag{66}$$

The transformation of $\mathbf{E}_\mathbf{z}^\mathbf{FF}$ in (65) into cylindrical coordinates is performed after substituting (66) into (11), in which gives,

$$\mathbf{P}_\mathbf{p}^\mathbf{2}\left(\mathbf{k}_\|\right) = \frac{k_{z2}}{k_2}\mathbf{x} - \frac{k_x}{k_2}\mathbf{z} = \boldsymbol{\theta}. \tag{67}$$

The time-averaged Poynting vector for perpendicular polarized dipole $\langle \mathbf{S}_\mathbf{z}^\mathbf{2D}\rangle$ is given by:

$$\left\langle \mathbf{S}_\mathbf{z}\left(\mathbf{r}\right)\right\rangle = \frac{1}{2z_2}\mathrm{Re}\left\{\mathbf{E}_\mathbf{z}^\mathbf{FF}\cdot\mathbf{E}_\mathbf{z}^\mathbf{FF*}\right\}\boldsymbol{\rho} = \frac{\omega^4\mu_1^2 k_2}{16\pi z_2\rho} p_z{}^2\cos^2\theta\left|T_p^z\right|^2\boldsymbol{\rho}, \tag{68}$$

The dipole's total radiated power (Q_z^{2D}) into medium 2 is then obtained integrating $\langle \mathbf{S}_\mathbf{z}^\mathbf{2D}\rangle$ along the upper semi-circle as follows,

$$Q_z = \int_{-\pi/2}^{\pi/2} \langle \mathbf{S_z}(\mathbf{r}) \rangle \cdot \boldsymbol{\rho} \rho \mathrm{d}\theta = \frac{\omega^4 \mu_1^2 k_2}{16\pi z_2} p_z^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta \left|T_p^z\right|^2 \mathrm{d}\theta. \tag{69}$$

2.3. Parallel polarized dipole

The dipole moment in this case is oriented along the x-axis, thus $\mathbf{p} = p_x\mathbf{x}$. After substituting $\mathbf{p}$ into (53) and (55), the following FTs of the reflected and transmitted electric fields are obtained,

$$\mathbf{E}_\mathbf{r}^\mathbf{FT}(k_x, z) = \mathrm{j}\frac{\omega^2 \mu_1}{\sqrt{8\pi}} p_x R_p^x(k_x, q) \mathbf{P}_\mathbf{p}^\mathbf{1}(k_x) e^{-\mathrm{j}k_{z1}(z-q)} \tag{70}$$

$$\mathbf{E}_\mathbf{t}^\mathbf{FT}(k_x, z) = \mathrm{j}\frac{\omega^2 \mu_1}{\sqrt{8\pi}} p_x T_p^x(k_x, z_{gr}) \mathbf{P}_\mathbf{p}^2(k_x) e^{\mathrm{j}k_{z1}(z-z_{gr})} \tag{71}$$

$$R_p^x(k_x, q) = \sum_{i=-\infty}^{\infty} \frac{1}{k_1} e^{\mathrm{j}k_{z1}^0 q} r_p^i(k_x^0), \tag{72}$$

$$T_p^x(k_x, z_{gr}) = \sum_{i=-\infty}^{\infty} \frac{1}{k_1} e^{\mathrm{j}k_{z1}^0 q} t_p^i(k_x^0). \tag{73}$$

2.3.1. Purcell factor

The dipole's total dissipated power is calculated after substituting (70) into (56) and the resulting equations into (25), which gives,

$$W = \frac{\omega^3 \mu_1}{8\pi} p_x^2 \operatorname{Re}\left\{ \int_{-\infty}^{\infty} \left[\frac{k_{z0}}{k_0^2} + R_x^z(k_x, q) \left(\mathbf{P}_\mathbf{p}^\mathbf{1}(k_x) \cdot \mathbf{x} \right) \mathrm{e}^{\mathrm{j}k_{z1}q} \right] \mathrm{d}k_x \right\}. \tag{74}$$

Therefore, the Purcell factor for a 2D parallel polarized dipole (P_x^{2D}) is obtained as follows,

$$P_x = \frac{W}{W_0} = 1 + \frac{2}{\pi} \operatorname{Re}\left\{ \int_{-\infty}^{\infty} R_p^x(k_x, q) \left[\mathbf{P}_\mathbf{p}^\mathbf{1}(k_x) \cdot \mathbf{x} \right] \mathrm{e}^{\mathrm{j}k_{z1}q} \mathrm{d}k_x \right\}. \tag{75}$$

2.3.2. Radiated power

The dipole's total radiated power is calculated by integrating the far field Poynting vector temporal mean over the upper semi-circle. To this end, we need to calculate the electric field generated by a parallel polarized dipole in the far field ($\mathbf{E}_\mathbf{x}^\mathbf{FF}$) which, according to the stationary phase method, is given by,

$$\mathbf{E}_x^\mathbf{FF}(\mathbf{r}) = -\mathrm{j}\frac{k_{z2}}{\sqrt{k_2}} E_t^{FT}(k_x, 0) \frac{\mathrm{e}^{\mathrm{j}k_2|\boldsymbol{\rho}|}}{\sqrt{|\boldsymbol{\rho}|}} \mathrm{e}^{-\mathrm{j}k_{z2}z_{gr}} = \frac{\omega^2 \mu_1}{\sqrt{8\pi}} p_x \sqrt{k_2} \cos\theta T_p^x \frac{\mathrm{e}^{\mathrm{j}k_2|\boldsymbol{\rho}|}}{\sqrt{|\boldsymbol{\rho}|}} \mathrm{e}^{-\mathrm{j}k_2 \cos\theta z_{gr}} \boldsymbol{\theta}, \tag{76}$$

Equation (76) gives the electric fields in cylindrical coordinates ($\boldsymbol{\rho}, \boldsymbol{\theta}$). The time-averaged Poynting vector and the dipole's total radiated power ($\langle \mathbf{S}_\mathbf{x}^\mathbf{2D} \rangle$ and Q_x^{2D}, respectively) are calculated as follows,

$$\langle \mathbf{S_x}(\mathbf{r}) \rangle = \frac{1}{2z_2} \operatorname{Re}\left\{ \mathbf{E}_\mathbf{x}^\mathbf{FF} \cdot \mathbf{E}_\mathbf{x}^\mathbf{FF*} \right\} \boldsymbol{\rho} = \frac{\omega^4 \mu_1^2 k_2}{16\pi z_2 \rho} p_x{}^2 \cos^2\theta \left|T_p^x\right|^2 \boldsymbol{\rho}, \tag{77}$$

$$Q_x = \int_{-\pi/2}^{\pi/2} \langle \mathbf{S_x}(\mathbf{r}) \rangle \cdot \boldsymbol{\rho} \rho \mathrm{d}\theta = \frac{\omega^4 \mu_1^2 k_2}{16\pi z_2} p_x^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta \left|T_p^x\right|^2 \mathrm{d}\theta, \tag{78}$$

3. Radiation pattern of the 3D HMM

Figure S3 shows the radiation patterns of the transmitted electric fields for p- (a,c) and s-polarization (b,c) emitted by a perpendicularly (a,b) and parallel (c,d) polarized dipole. On one hand, perpendicularly polarized dipoles do not radiate *s*-polarized waves, therefore the *s*-polarized electric field shown in Fig. S3 (b) is generated by the cross-polarization terms in (18). The peaks in the radiation pattern for *p*-polarized waves (Fig. S3 (a)) are inside the 20° cone, while the peaks for *s*-polarized waves are in the 55° cone (Fig. S3 (b)). On the other hand, parallel polarized dipoles radiate both *p*- and *s*-polarized waves. However, *s*-polarized waves are highly attenuated by the HMM, and consequently do not have a considerable influence on the transmitted fields. As a result, the *s*-polarized waves are mostly generated by the cross-polarization term $t_{s,p}^{i,m}$ in (16). Note once again that the peaks of the *p*-polarized waves (Fig. S3 (c)) are in a smaller cone compared to the *s*-polarized waves (Fig. S3 (d)) (10° and 30° for *p*- and *s*-polarized waves, respectively).

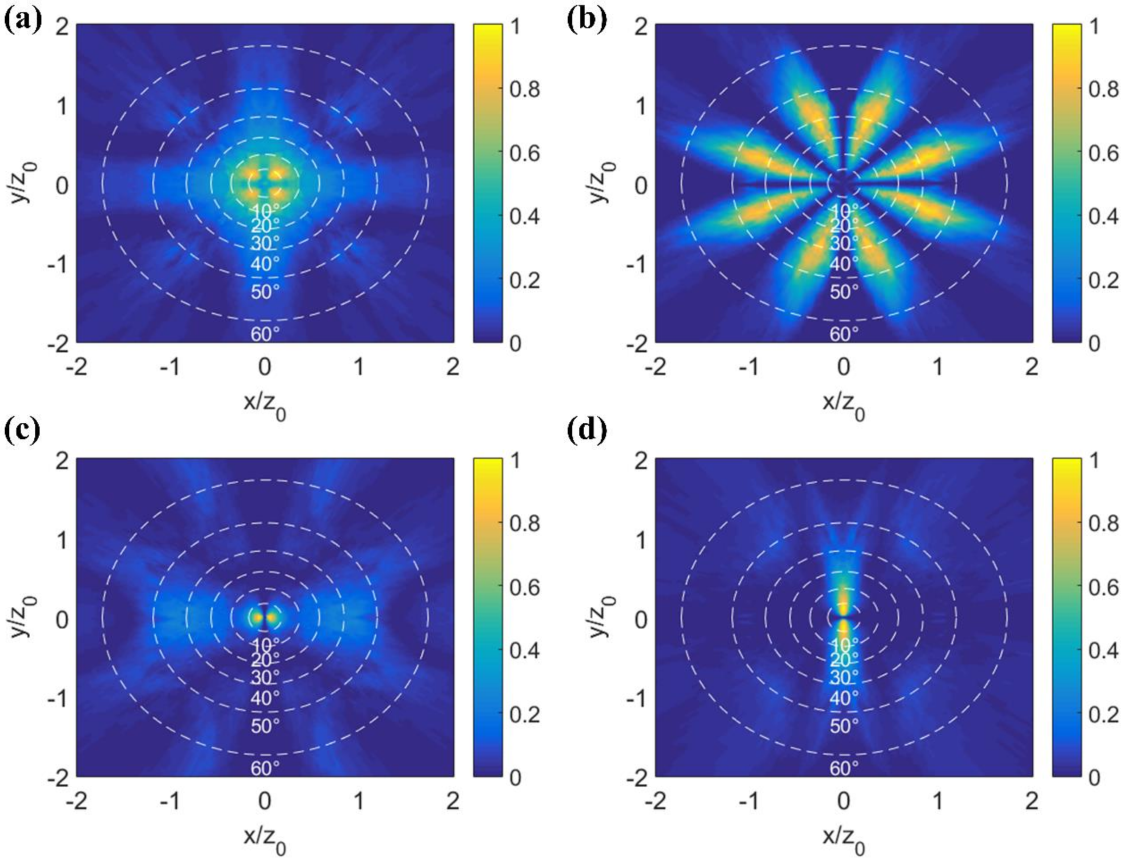

Figure S3. Radiation pattern of the transmitted electric fields emitted by a perpendicularly (a,b) and parallel (c,d) polarized dipole decomposed in *p*- (a,c) and *s*-polarization (b,d).

4. Lumerical FDTD Simulation

The HMM with the metallic gratings on top were drown in Lumerical FDTD, as shown in Fig. S3. The description of the simulation is listed below:

- Perfect Matched Layers (PML) for the surround the boundary condition.
- The total simulation area is 4µm x 4µm and the mesh inside the structure is Δx=Δy=0.5nm.
- Taking advantage of the symmetry of the simulations, we utilized the Asymmetric and Symmetric boundary condition for parallel or perpendicular dipole, respectively.
- A broadband (673-873nm) dipole source centered at λ = 773nm was placed 10 nm under the HMM.
- Two vertical monitors were placed above the gratings in order to calculate the radiation pattern in the far-field.
- By integrating the Poynting vector over the lines that surround the simulation area above the metallic gratings we calculated the power radiated to the far field.
- The calculation of the Purcell factor was performed by the Lumerical FDTD.

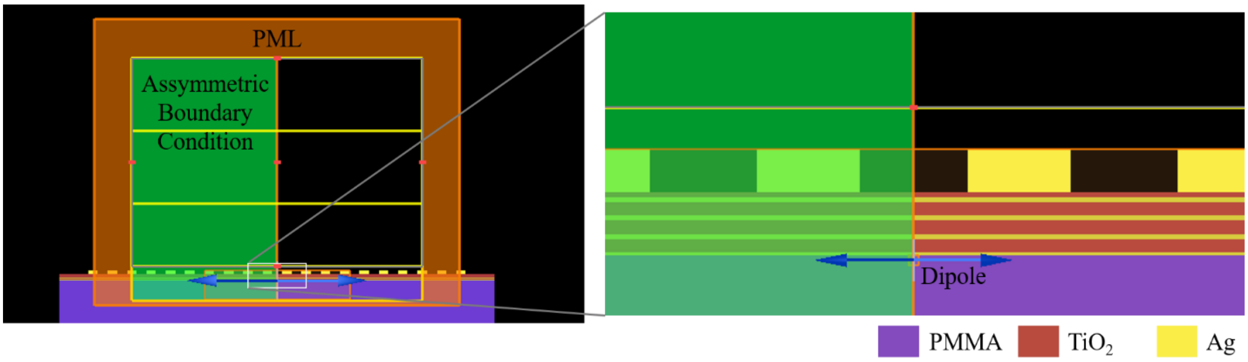

Figure S4. Schematic of the Lumerical 2D-FDTD simulation.

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